Transformation Structures for 2-Group Actions in Higher Gauge Theory

(Joint work with Roger Picken)

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Original motivation

Certain TQFT's can be factored as

$$nCob \xrightarrow{F} Span(Gpd) \xrightarrow{D} Vect \tag{1}$$

and may often be promoted to Extended TQFT via

$$nCob_2 \xrightarrow{F} Span_2(Gpd) \xrightarrow{\Lambda} 2Vect \tag{2}$$

The functor D and 2-functor Λ are Baez-Dolan *degroupoidification* and *2-linearization* resepectively, and are general. Different choices of groupoid-valued functors F for manifolds then give different theories.
The DW model uses $F = A_0^G$, which assigns to a manifold M the groupoid of flat G-bundles over M.

Our Motivating Example

Our original motivation in developing this machinery was to describe the **symmetry** of **moduli spaces** of **connections on gerbes**.

There is a handy fact about connections on bundles:

$$Fun(\Pi_1(M), G) \cong A_0^G(M) = Conn(M) /\!\!/ Gauge(M)$$

where $Conn(M)$ is a *moduli space* of connections, $Gauge(M)$ is a total group of *gauge transformations* which acts on $Conn(M)$, and we have here the *weak quotient* by this action.

For 1-groups, both of these are groupoids. For 2-groups, the analogous objects are a 2-groupoid and a double-groupoid respectively!

Question: Is there a corresponding equvalence?

2-Groups and Crossed Modules

Idea: *Categorification of symmetry*. To describe symmetry of categories, generalize group actions.

Definition

A **2-group** $\mathcal{G}$ is a 2-category with one object, and all morphisms and 2-morphisms invertible.

A **categorical group** is a group object in **Gpd**: a category $\mathcal{G}$ with $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and an inverse map satisfying the usual group axioms. In particular, a monoidal category $(G, \otimes)$ where every object and morphism is invertible with respect to $\otimes$.

I will use "2-group" for both except where the distinction is critical.

2-groups are classified by crossed modules:

Definition

A crossed module consists of $(G, H, \triangleright, \partial)$, where:

- G, $H \in \mathbf{Grp}$
- $G \triangleright H$ an action of G on H by automorphisms
- $\partial : H \to G$ a homomorphism

satisfying

- $\partial(g \triangleright h) = g\partial(h)g^{-1}$
- $\partial(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}$

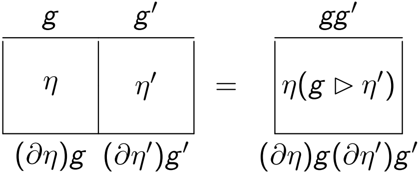
A homomorphism of crossed modules is a pair of homomorphisms $G \to G'$ and $H \to H'$ which is compatible with $\triangleright$.

The category of crossed modules and homomorphisms is equivalent to the category of 2-groups under with a correspondence determined by $\mathbf{G}(G, H, \triangleright, \partial)$ with:

- **Objects**: elements of G
- **Morphisms**: elements of the semidirect product $G \ltimes H$, with $(g, h) : g \to (\partial h)g$ and

$$(\partial(\eta)g, \zeta) \circ (g, \eta) = (g, \zeta\eta).$$

- **Monoidal Structure**:



Definition

A (strict) action of a 2-group $\mathcal{G}$ on a category $\mathbf{C}$ is a functor $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ (strictly) satisfying the action diagram in $\mathbf{Cat}$:

$$\begin{array}{ccc}
\mathcal{G} \times \mathcal{G} \times \mathbf{C} & \xrightarrow{\otimes \times Id_{\mathbf{C}}} & \mathcal{G} \times \mathbf{C} \\
{\scriptstyle Id_{\mathcal{G}} \times \hat{\Phi}} \downarrow & & \downarrow {\scriptstyle \hat{\Phi}} \\
\mathcal{G} \times \mathbf{C} & \xrightarrow[\hat{\Phi}]{} & \mathbf{C}
\end{array}$$

Actions of 2-groups make sense in any 2-category, but only take this special form (by *currying*) in $\mathbf{Cat}$:

Lemma

For any $\mathbf{C} \in \mathbf{Cat}$*, a strict monoidal functor* $\Phi : \mathcal{G} \to End(\mathbf{C})$ *is equivalent to a strict action* $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$*.*

Three Group Actions in a 2-Group Action

If $\mathcal{G}$ is a 2-group classified by the crossed module $(G, H, \triangleright, \partial)$, and $\hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$ is a strict action, by abuse of notation we denote by $\blacktriangleright$ three interconnected group actions. Two actions of G on objects and morphisms of $\mathbf{C}$:

- Given $\gamma \in \mathcal{G}^{(0)} = G$ and $x \in \mathbf{C}^{(0)}$, let
$$\gamma \blacktriangleright x = \Phi_\gamma(x) = \hat{\Phi}(\gamma, x)$$
- Given $\gamma \in \mathcal{G}^{(0)} = G$ and $f \in \mathbf{C}^{(1)}$, let
$$\gamma \blacktriangleright f = \Phi_\gamma(f) = \hat{\Phi}((\gamma, 1_H), f)$$

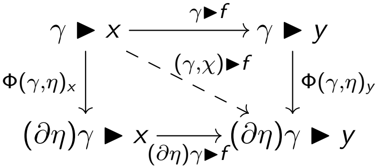
(Which are compatible with the structure maps of $\mathbf{C}$: source, target, composition, etc.)

One action of $G \ltimes H$ on morphisms of $\mathbf{C}$:

- Given $(\gamma, \chi) \in \mathcal{G}^{(1)} = G \ltimes H$ and $(f : x \to y) \in \mathbf{C}^{(1)}$, let

$$\begin{aligned} (\gamma, \chi) \blacktriangleright f &= \hat{\Phi}((\gamma, \chi), f) \\ &= \Phi_{(\gamma,\chi)}(y) \circ (\gamma \blacktriangleright f) \\ &= (\partial(\chi)\gamma \blacktriangleright f) \circ \Phi_{(\gamma,\chi)}(x) \end{aligned}$$

This is the diagonal of the naturality square associated to $f : x \to y$ in $\mathbf{C}$:



(Note $\Phi_{(\gamma,\chi)}$ typically assigns a nonidentity morphism to x, so there is no action of $G \ltimes H$ on objects of $\mathbf{C}$)

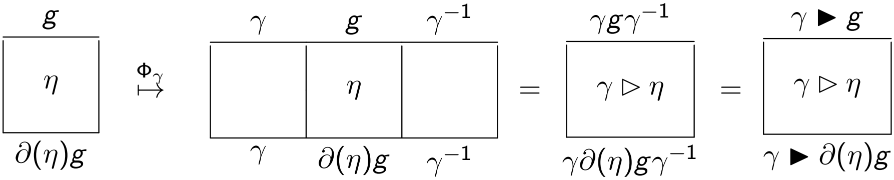
Example: Adjoint Action of $\mathcal{G}$

Definition (Part 1)

If $\mathcal{G} \sim (G, H, \triangleright, \partial)$, then the *adjoint action* of $\mathcal{G}$ (seen as a 2-group) on $\mathcal{G}$ (seen as a categorical group) is given by the 2-functor:

- $\gamma \in Ob(\mathcal{G})$ gives an endofunctor $\Phi_\gamma : \mathcal{G} \to \mathcal{G}$ which itself is defined by:
  - $\Phi_\gamma(g) = \gamma g \gamma^{-1}$
  - $\Phi_\gamma(g, \eta) = (\gamma g \gamma^{-1}, \gamma \triangleright \eta)$

Draw this as:



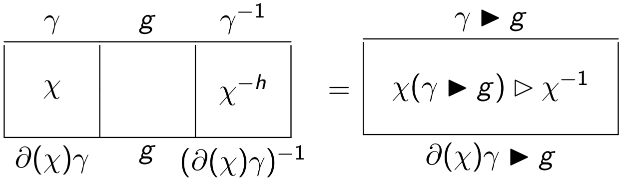
Definition (Part 2)

- $(\gamma, \chi) \in Mor(\mathcal{G})$ gives a natural transformation $\Phi_{(\gamma,\chi)} : \Phi_\gamma \Rightarrow \Phi_{\partial(\chi)\gamma}$, defined by:

$$\Phi_{(\gamma,\chi)}(g) = (\gamma g \gamma^{-1}, \chi(\gamma g \gamma^{-1}) \triangleright \chi^{-1}))$$

Draw this as:

$\Phi_{(\gamma,\chi)}(g) =$



Transformation Groupoids

Global symmetry involves actions. *Local symmetry relations* of a set X is represented as a groupoid (in **C**) with:

- Objects: the elements of X
- Isomorphisms: $f : x \to y$ denoting a symmetry relation between x and y

Any global symmetry action gives a local symmetry groupoid (but not necessarily conversely):

Definition

The **transformation groupoid** of an action of a group G on a set X is the groupoid $X /\!\!/ G$ with:

- **Objects**: X (that is, all $x \in X$)
- **Morphisms**: $G \times X$, with $(g, x) : x \mapsto g \rhd x$
- **Composition**: (Multiplication in G)

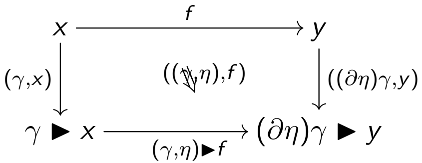
Transformation Double Categories

Definition

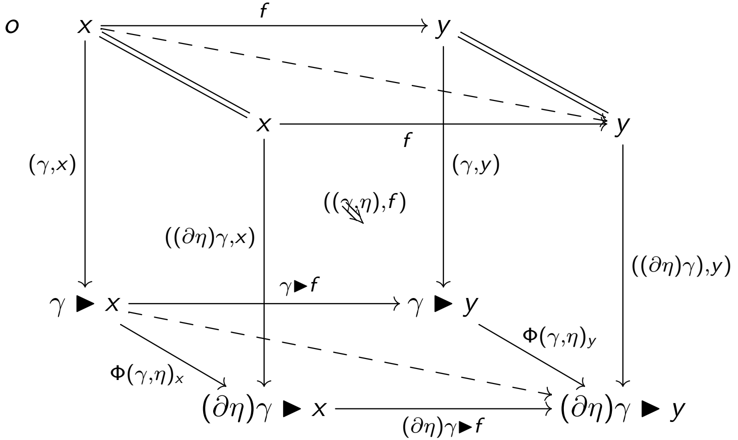
Given a 2-group $\mathcal{G}$, a category $\mathbf{C}$, and an action of $\mathcal{G}$ on $\mathbf{C}$, the transformation 2-groupoid $\mathbf{C}/\!\!/\mathcal{G}$ is the groupoid in **Cat** with:

- $Ob(\mathbf{C}/\!\!/\mathcal{G}) = \mathbf{C}$.
- $Mor(\mathbf{C}/\!\!/\mathcal{G}) = \mathcal{G} \times \mathbf{C}$, with
  - **Source** functor $s = \pi_{\mathbf{C}} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$
  - **Target** functor $t = \hat{\Phi} : \mathcal{G} \times \mathbf{C} \to \mathbf{C}$
  - **Composition**: (given by the action condition)

This is really a *double category* - the morphisms of $Mor(\mathbf{C}/\!\!/\mathcal{G})$ are squares:



The squares describe the action of $G \ltimes H$ on $Mor(\mathbf{C})$. They come from diagonals of the cube:



One way to collect all (flat) $\mathcal{G}$-connections on gerbes over a manifold M is as the *2-groupoid* (!) of transport 2-functors:

$$2Fun(\Pi_2(M), \mathcal{G}) \tag{3}$$

where $\Pi_2(M)$ is the **fundamental 2-groupoid** of M consisting of

- **Objects**: $x \in M$
- **Morphisms**: Paths $I \to M$
- **2-Morphisms**: Homotopies $I^2 \to M$ fixing endpoints (up to homotopy)

Then $2Fun(\Pi_2(M), \mathcal{G})$ has:

- **Objects**: 2-functors from $\Pi_2(M)$ to $\mathcal{G}$
- **Morphisms**: Pseudonatural transformations between 2-functors
- **2-Morphisms**: Modifications

Definition

A *pseudonatural transformation* $p : F \Rightarrow G$ between 2-functors assigns data filling the following square:

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ {\scriptstyle p_x}\downarrow & \swarrow\!\!\!\!\swarrow {\scriptstyle p_f} & \downarrow{\scriptstyle p_y} \\ G(a) & \xrightarrow[G(f)]{} & G(b) \end{array}$$

compatible with composition and units, and such that for all 2-morphisms $\alpha : f \to g$ in $\mathbf{A}$, a certain 3D diagram commutes. We say p is *strict* if all $p(f) = Id$ (i.e. a "natural transformation"), and *costrict* if for all $x \in \mathbf{A}$, we have $p(x) \equiv Id_{F(x)}$ (thus in particular, $F(x) = G(x)$). (i.e. an "ICON")

Lemma

*If **A** and **B** are bigroupoids, and F, G are functors from **A** to **B**, then every pseudonatural transformation $p : F \Rightarrow G$ is a composite of a strict transformation followed by a costrict transformation. Such p is also a composite of a costrict transformation followed by a strict transformation:*

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
{\scriptstyle n(x)}\downarrow & \overset{q(f)\circ 1_{n(x)}}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle n(y)} \\
G(x) & \xrightarrow[G(f)]{} & G(y)
\end{array}
\quad = \quad
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
{\scriptstyle n(x)}\downarrow & & \downarrow{\scriptstyle n(y)} \\
G'(x) & \xrightarrow{G'(f)} & G'(y) \\
\Vert & \overset{q(f)}{\swarrow\!\!\!\!\swarrow} & \Vert \\
G(x) & \xrightarrow[G(f)]{} & G(y)
\end{array}
\tag{4}
$$

(And similarly in the reverse order)

Definition

If **A** and **B** are bicategories, define the double category of 2-functors:

$$Hom_{\square}(\mathbf{A},\mathbf{B}) = DblCat(V(\mathbf{A}), V(\mathbf{B})) \tag{5}$$

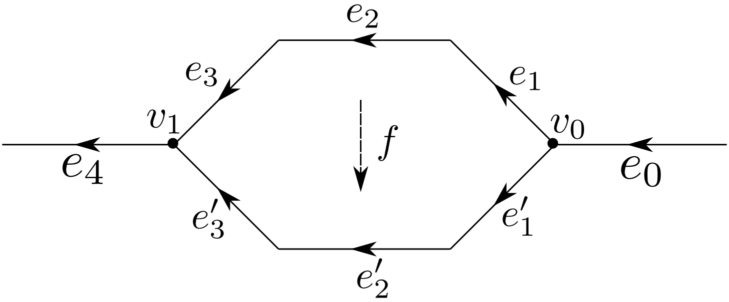
If **A** and **B** and bigroupoids, the data of the double groupoid $Hom_{\square}(\mathbf{A},\mathbf{B})$ are in 1-1 correspondence with:

- **Objects**: Strict 2-functors from **A** to **B**
- **Horizontal Morphisms**: Costrict pseudonatural transformations
- **Vertical Morphisms**: Strict natural transformations
- **Squares**: Squares correspond to modifications are modifications $M : s_2 \circ c_F \Rightarrow c_G \circ s_1$:

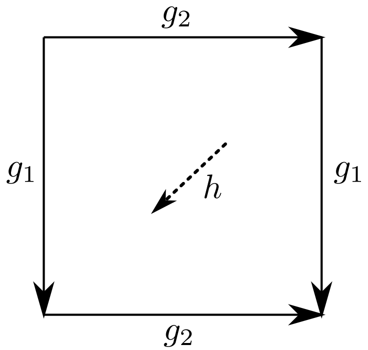
$$\begin{array}{ccc} F_1 & \xrightarrow{c_F} & F_2 \\ {\scriptstyle s_1}\downarrow & \swArrow_{M} & \downarrow{\scriptstyle s_2} \\ G_1 & \xrightarrow[c_G]{} & G_2 \end{array} \tag{6}$$

Category of Connections

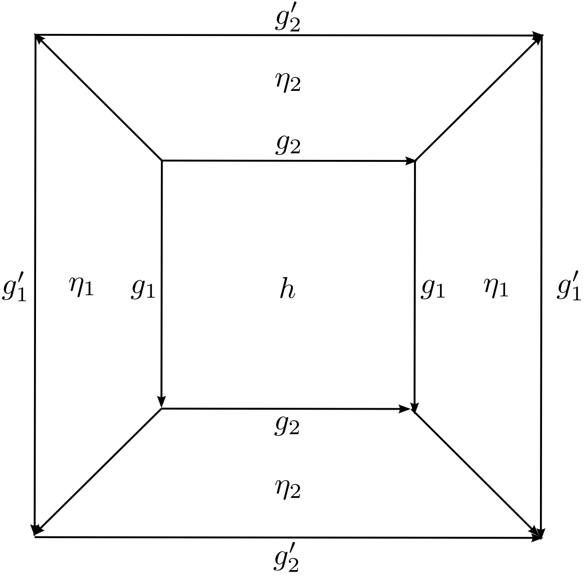
Think of connections on gerbes as determining **holonomies** valued in a 2-group $\mathcal{G} \sim (G, H, \triangleright, \partial)$ to paths and homotopies of paths in a manifold.
(Note: from fields of Lie-group valued forms, one can get these by integration).
We simplified things by using discrete paths and homotopies made of chosen edges and faces:



Given gauge 2-group $\mathcal{G}$, the objects of **Conn** must be connections - which assign G-holonomies to edges and H-holonomies to faces:



A Connection on $(T^2, \mathcal{D})$
Morphisms assign H-holonomies to edges...



Morphism in **Conn**$(T^2, \mathcal{D})$

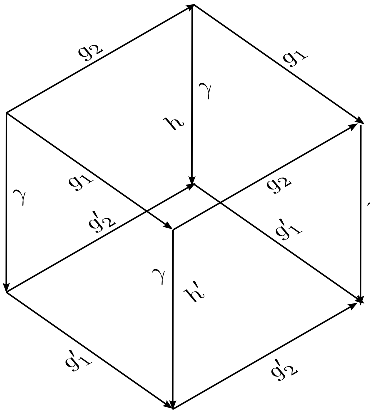
Then indeed:

Theorem
Given a manifold (M)*, and a strict 2-group* $\mathcal{G}$ *presented by the crossed module* $(G, H, \triangleright, \partial)$*, there is an isomorphism:*

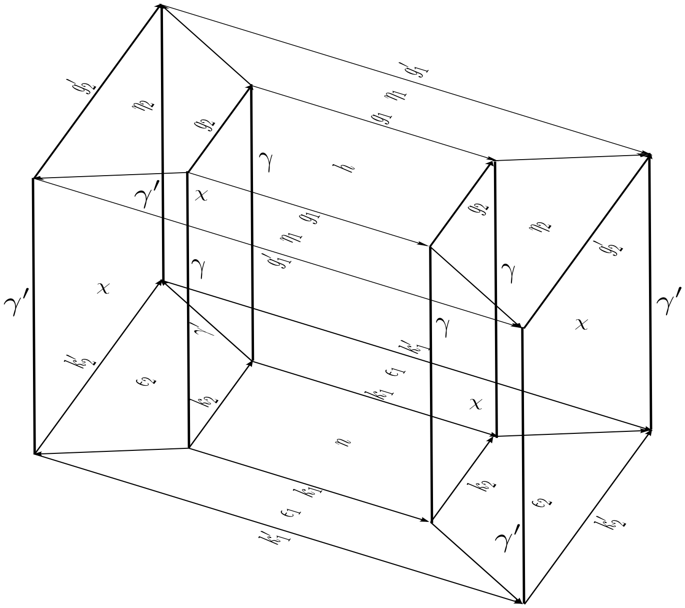
$$\mathbf{Conn}/\!\!/\mathbf{Gauge} \cong Hom_{\square}(\Pi_2(M), \mathcal{G}) \tag{7}$$

Where **Conn** is a category of connections and **Gauge** is a 2-group best understood as $\mathcal{G}^M$ (or $\mathcal{G}^V$ in our discrete case, where V is the set of vertices in a cell structure on M).

Note: this means that *some local symmetry lives in* **Conn** *to begin with, and only some comes from the global action*.



Vertical Morphism in **Conn**$/\!\!/$**Gauge**$(T^2, \mathcal{D})$ (From action of G^V)



Square - Gauge Modification on $(T^2, \mathcal{D})$ (from action of $(G \ltimes H)^V$)