Graph Homologies and Functoriality

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Goal

Open a door between Graph(knot) homology and applications in Dynamical Systems and Noncommutative Geometry.
Tools:

- Additive category (Obj Embedded graphs in the 3-sphere, Mor geometric correspondences given by 3-manifold M branched coverings of the 3-sphere along embedded graphs (or in particular knots) in the 3-sphere).
- Kauffman's invariant of Graphs.
- Khovanov Homology for graphs and Floer Homology for Graphs.
- PL Cobordisms between graphs and Smooth Cobordisms between family of links.

1. We construct an additive category where objects are embedded graphs in the 3-sphere and morphisms are geometric correspondences given by 3-manifolds realized in different ways as branched covers of the 3-sphere, up to branched cover cobordisms, by considering a 3-manifold M realized in two different ways as a covering of the 3-sphere as a correspondence between the branch loci (Graphs) of the two covering maps.

$$G \subset S^3 \xleftarrow{\pi_G} M \xrightarrow{\pi_{G'}} S^3 \supset G'$$

2. We consider dynamical systems obtained from associated convolution algebras endowed with time evolutions defined in terms of the underlying geometries.
3. We describe the relevance of our construction to the problem of spectral correspondences in noncommutative geometry.

Example

(Poincaré homology sphere): Let **P** denote the Poncaré homology sphere. This smooth compact oriented 3-manifold is a 5-fold cover of S^3 branched along the *trefoil knot* (that is, the $(2,3)$ torus knot), or a 3-fold cover of S^3 branched along the $(2,5)$ torus knot, or also a 2-fold cover of S^3 branched along the $(3,5)$ torus knot.

$\mathcal{K}$ denote the category whose objects $Obj(\mathcal{K})$ are graphs $G \subset S^3$ and whose morphisms $Mor(\mathcal{K})$ a 3-manifold M_i with submersions π_G and $\pi_{G'}$ to S^3

The composition $M \circ \tilde{M}$

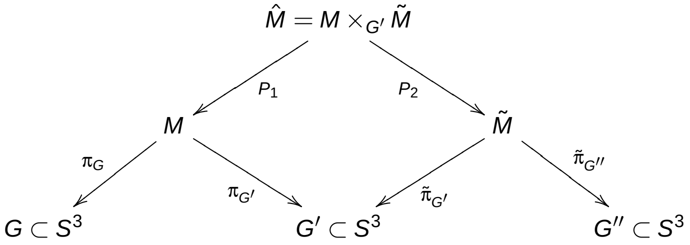
Definition

Suppose given

$$G \subset S^3 \xleftarrow{\pi_G} M \xrightarrow{\pi_{G'}} S^3 \supset G' \quad \text{and} \quad G' \subset S^3 \xleftarrow{\tilde{\pi}_{G'}} \tilde{M} \xrightarrow{\tilde{\pi}_{G''}} S^3 \supset G''.$$

One defines the composition $M \circ \tilde{M}$ as fibered product $M \times_{G'} \tilde{M}$

$$M \circ \tilde{M} := M \times_{G'} \tilde{M} = \{(x, x') \in M \times \tilde{M} | \pi_{G'}(x) = \tilde{\pi}_{G'}(x')\}.,$$



Cobordism of branched cover by Hilden and Little 1980

- Two compact oriented 3-manifolds M_1 and M_2 that are branched covers of S^3, with covering maps $\pi_1 : M_1 \to S^3$ and $\pi_2 : M_2 \to S^3$, respectively branched along 1-dimensional simplicial complex E_1 and E_2.
- A cobordism of branched coverings is a 4-dimensional manifold W with boundary $\partial W = M_1 \cup -M_2$ (where the minus sign denotes the change of orientation), endowed with a submersion $q : W \to S^3 \times [0,1]$, with $M_1 = q^{-1}(S^3 \times \{0\})$ and $M_2 = q^{-1}(S^3 \times \{1\})$ and $q|_{M_1} = \pi_1$ and $q|_{M_2} = \pi_2$.
- The map q is a covering map branched along a surface $S \subset S^3 \times [0,1]$ such that $\partial S = E_1 \cup -E_2$, with $E_1 = S \cap (S^3 \times \{0\})$ and $E_2 = S \cap (S^3 \times \{1\})$.
- Two morphisms M_1 and M_2 in $Hom(G, G')$, of the form

$$G \subset E_1 \subset S^3 \xleftarrow{\pi_{G,1}} M_1 \xrightarrow{\pi_{G',1}} S^3 \supset E'_1 \supset G'$$

$$G \subset E_2 \subset S^3 \xleftarrow{\pi_{G,2}} M_2 \xrightarrow{\pi_{G',2}} S^3 \supset'_2 \supset G'.$$

$$S \subset S^3 \times [0,1] \xleftarrow{q} W \xrightarrow{q'} S^3 \times [0,1] \supset S',$$

branched along surfaces $S, S' \subset S^3 \times [0,1]$. The maps q and q' have the properties that $M_1 = q^{-1}(S^3 \times \{0\}) = q'^{-1}(S^3 \times \{0\})$ and $M_2 = q^{-1}(S^3 \times \{1\}) = q'^{-1}(S^3 \times \{1\})$, with $q|_{M_1} = \pi_{G,1}$, $q'|_{M_1} = \pi_{G',1}$, $q|_{M_2} = \pi_{G,2}$ and $q'|_{M_2} = \pi_{G',2}$. The surfaces S and S' have boundary $\partial S = E_1 \cup -E_2$ and $\partial S' = E_1' \cup -E_2'$, with $E_1 = S \cap (S^3 \times \{0\})$, $E_2 = S \cap (S^3 \times \{1\})$, $E_1' = S' \cap (S^3 \times \{0\})$, and $E_2' = S' \cap (S^3 \times \{1\})$. Here By “surface" we mean a 2-dimensional simplicial complex that is PL-embedded in $S^3 \times [0,1]$, with boundary $\partial S \subset S^3 \times \{0,1\}$ given by 1-dimensional simplicial complexes, *i.e.* embedded graphs.

Convolution algebra and time evolution

Lemma

The set of compact oriented 3-manifolds $\mathcal{G}$ forms a regular semigroupoid, whose set of units is identified with the set of embedded graphs.

Consider the semigroupoid ring (algebra) $\mathbb{C}[\mathcal{G}]$ of complex valued functions with finite support on $\mathcal{G}$, with the associative convolution product,

$$(f_1 * f_2)(M) = \sum_{M_1, M_2 \in \mathcal{G} : M_1 \circ M_2 = M} f_1(M_1) f_2(M_2).$$

Define an involution on the semigroupoid $\mathcal{G}$ by

$$Hom(G, G') \ni \alpha = (M, G, G') \mapsto \alpha^\vee = (M, G', G) \in Hom(G', G),$$

where, if α corresponds to the 3-manifold M with branched covering maps then $\alpha^\vee$ corresponds to the same 3-manifold taken in the opposite order.

Lemma

The algebra $\mathbb{C}[\mathcal{G}]$ is an involutive algebra with the involution

$$f^\vee(M) = \overline{f(M^\vee)}.$$

Time evolution

Given an algebra $\mathcal{A}$ over $\mathbb{C}$, a time evolution is a 1-parameter family of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$. There is a natural time evolution on the algebra $\mathbb{C}[\mathcal{G}]$ obtained as follows.

Lemma

Suppose given a function $f \in \mathbb{C}[\mathcal{G}]$. Consider the action defined by

$$\sigma_t(f)(M) := \left(\frac{n}{m}\right)^{it} f(M),$$

where M a covering with the covering maps π_G and $\pi_{G'}$ respectively of generic multiplicity n and m. This defines a time evolution on $\mathbb{C}[\mathcal{G}]$.

Given a representation $\rho : \mathcal{A} \to \mathrm{End}(\mathcal{H})$ of an algebra $\mathcal{A}$ with a time evolution σ, one says that the time evolution, in the representation ρ, is generated by a Hamiltonian H if for all $t \in \mathbb{R}$ one has

$$\rho(\sigma_t(f)) = e^{itH} \rho(f) e^{-itH},$$

for an operator $H \in \mathrm{End}(\mathcal{H})$.

Convolution algebras and 2-semigroupoids

Lemma

The data of embedded graphs in the 3-sphere, 3-dimensional geometric correspondences, and 4-dimensional branched cover cobordisms form a 2-category $\mathcal{G}^2$.

We denote the compositions of 2-morphisms by the notation

$$\text{horizontal (fibered product): } W_1 \circ W_2 \qquad \text{vertical (gluing): } W_1 \bullet W_2.$$

Vertical and horizontal time evolutions

We obtain a convolution algebra associated to the 2-semigroupoid This space of functions can be made into an algebra $\mathcal{A}(\mathcal{G}^2)$ with the associative convolution product of the form

$$(f_1 \bullet f_2)(W) = \sum_{W = W_1 \bullet W_2} f_1(W_1) f_2(W_2),$$

which corresponds to the vertical composition of 2-morphisms, namely the one given by the gluing of cobordisms. Similarly, one also has on $\mathcal{A}(\mathcal{G}^2)$ an associative product which corresponds to the horizontal composition of 2-morphisms, given by the fibered product of cobordisms, of the form

$$(f_1 \circ f_2)(W) = \sum_{W = W_1 \circ W_2} f_1(W_1) f_2(W_2).$$

We also have an involution compatible with both the horizontal and vertical product structure.

Vertical and horizontal time evolutions

We say that σ_t is a *vertical time evolution* on $\mathcal{A}(\mathcal{G}^2)$ if it is a 1-parameter group of automorphisms of $\mathcal{A}(\mathcal{G}^2)$ with respect to the product structure given by the vertical composition of 2-morphisms namely

$$\sigma_t(f_1 \bullet f_2) = \sigma_t(f_1) \bullet \sigma_t(f_2).$$

Similarly, a *horizontal time evolution* on $\mathcal{A}(\mathcal{G}^2)$ satisfies

$$\sigma_t(f_1 \circ f_2) = \sigma_t(f_1) \circ \sigma_t(f_2).$$

- Vertical time evolution: Hartle-Hawking gravity.
- Vertical time evolution: gauge moduli and index theory.
- Horizontal time evolution: bivariant Chern character.

Kauffman's invariant of Graphs 1989

1. Let G be a graph embedded in a 3-manifold M.
2. Associate to G a family of knots and links prescribes that we should make a local replacement as in figure to each vertex in G.
3. A vertex v connects two edges and isolates all other edges at that vertex, leaving them as free ends.

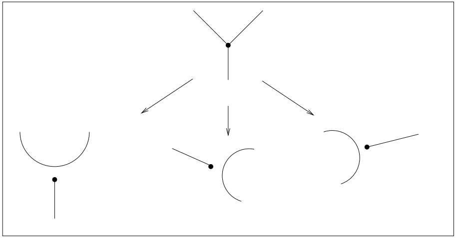


Figure: local replacement to a vertex in the graph G

Define $T(G)$ to be the family of the links associated to the graph G.

Theorem

Let G be any graph embedded in S^3, and presented diagrammatically. Then the family of knots and links $T(G)$, taken up to ambient isotopy, is a topological invariant of G.

For example, in the figure the graph G_2 is not ambient isotopic to the graph G_1, since $T(G_2)$ contains a non-trivial link.

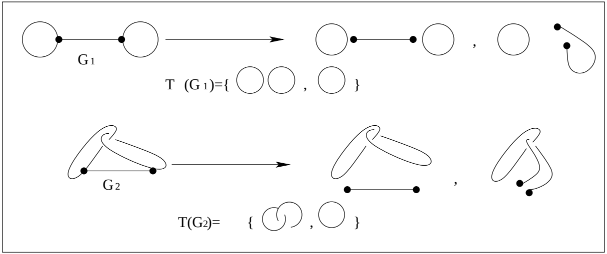


Figure: family of knots and links

Khovanov-Kauffman Homology for Graphs

(Khovanov 2000)

$$D \overset{Khovanov}{\longrightarrow} [D] = C^{*,*}(D) \overset{Homology}{\longrightarrow} KH^{*,*}(D)$$

Let G be an embedded graph, $T(G) = \{L_1, L_2, ..., L_n\}$ the family of links associated to G Then the Khovanov-Kauffman homology for the embedded graph G is given by

$$KKh(G) = Kh(L_1) \oplus Kh(L_2) \oplus .... \oplus Kh(L_n)$$

Example

In figure $T(G_1) = \{\bigcirc\bigcirc, \bigcirc\}$

$$KKh(G_1) = Kh(\bigcirc\bigcirc) \oplus Kh(\bigcirc)$$

$$KKh(G_1) = Kh(\bigcirc) \otimes Kh(\bigcirc) \oplus Kh(\bigcirc)$$

$T(G_2) = \{$  $, \bigcirc\}$ then

$KKh(G_2) = Kh($  $) \oplus Kh(\bigcirc)$

Floer-Kauffman Homology for Knots

Ozsváth - Szabó and Rasmussen around 2003 Introduced Floer Homology which is an invariant of knots and links in three manifolds. Let $K \subset S^3$ be an oriented knot, there are several different variants of the knot Floer homology of K.
The simplest is the hat version, which takes the form of a bi-graded, finitely generated Abelian group

$$\widehat{HFK}(K) = \bigoplus_{i,s\in\mathbb{Z}} \widehat{HFK}_i(K,s)$$

Here, i is called the Maslov (or homological) grading, and s is called the Alexander grading. The graded Euler characteristic of $\widehat{HFK}$ is the Alexander-Conway polynomial

$$\sum_{s,i\in\mathbb{Z}} (-1)^i q^s rank_{\mathbb{Z}}(\widehat{HFK}_i(K,s)) = \triangle_K(t)$$

Example

We used kauffman technique to introduce Khovanov-Kauffman homology for embedded graphs.

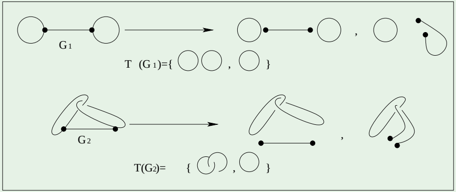


Figure: Family of links associated to a graph

$T(G_1) = \{\bigcirc\bigcirc, \bigcirc\}$

$$\widehat{HFG}(G_1) = \widehat{HFK}(\bigcirc\bigcirc) \oplus \widehat{HFK}(\bigcirc)$$

Now,

$$\widehat{HFG}(G_1) = \widehat{HFK}(\bigcirc) \otimes \widehat{HFK}(\bigcirc) \oplus \widehat{HFK}(\bigcirc)$$

Example

$T(G_2) = \{$  ,  $\}$ then

$$\widehat{HFG}(G_2) = \widehat{HFK}($$  $$) \oplus \widehat{HFK}($$  $$)$$

Graph Cobordance

Definition

Two links L and L' are called cobordic if there is a surface Σ have the boundary $\partial\Sigma = L \cup -L'$ with $L = \Sigma \cap (S^3 \times \{0\})$, $L' = \Sigma \cap (S^3 \times \{1\})$. Here by "surfaces" we mean 2-dimensional compact differentiable manifold embedded in $S^3 \times [0,1]$. We define the identity cobordism to be id_L for a link L. $[\Sigma_L]$ *denotes the cobordism class of the link L.*

Definition

Two graphs G and G' are called cobordic if there is a PL surface Σ_P have the boundary $\partial\Sigma_P = G \cup -G'$ with $G = \Sigma \cap (S^3 \times \{0\})$, $G' = \Sigma \cap (S^3 \times \{1\})$. Here by "surfaces" we mean 2-dimensional simplicial complexes that are PL-embedded in $S^3 \times [0,1]$. We define the identity cobordism to be id_G for a graph G. $[\Sigma_P]$ *denotes the cobordism class of the graph G.*

Definition

Let G, G' two embedded graphs have associated families of links $T(G)$ and $T(G')$ respectively according to Kauffman construction, and let $T(\Sigma_\alpha)_{\alpha\in\lambda}$ be a family of smooth cobordisms have links boundaries of $T(G)$ and $T(G')$. These graphs are said to be cobordant if there is a family of smooth cobordisms $T(\Sigma_\alpha)_{\alpha\in\lambda}$ in $S^3 \times [0,1]$ such that $T(G)$ and $T(G')$ are the boundary of $T(\Sigma_\alpha)$.

Theorem

The family of smooth cobordisms $T(\Sigma_\alpha)_{\alpha\in\lambda}$ is associated to the PL cobordism Σ_P.

Composition of Cobordisms

- We can define the composition between two PL cobordisms Σ_P and Σ'_P (where Σ_P is the cobordism with boundary $G \cup -G'$, Σ'_P is the cobordism with boundary $G' \cup -G''$ in $S^3 \times [0,1]$) to be $\Sigma_P \circ \Sigma'_P$ which is a PL cobordism with boundary $G \cup -G''$.
- For the second type of the cobordance by family of smooth cobordisms we can define the composition as follows : Let G, G' and G'' be embedded graphs in S^3 with $T(G), T(G')$ and $T(G'')$ links families associated to each graph respectively.
- We have a family of smooth cobordisms $T(\Sigma_\alpha)_{\alpha\in\lambda}$ have boundary $T(G) \cup -T(G')$ and another family $T(\Sigma'_\beta)_{\beta\in\gamma}$ with boundary $T(G') \cup -T(G'')$. Let Σ_β and $\alpha \in \lambda$ be a smooth cobordism with boundary links from the sets $T(G)$ and $T(G')$ and let Σ'_β for $\beta \in \gamma$ be a smooth cobordism with boundary links from the sets $T(G')$ and $T(G'')$. $\Sigma_\alpha \circ \Sigma'_\beta$ is a smooth cobordism with boundary links from the sets $T(G)$ and $T(G'')$ and this defines the composition of the second type of the graphs cobordance.

Let $\mathcal{G}_{\mathcal{K}}$ be a category, whose objects are embedded graphs that have a family of links according to the kauffman definition and morphisms are 2-dimensional simplicial complex surface $\Sigma_P \in Hom(G, G')$ with graphs boundary.

Lemma

$\mathcal{G}_{\mathcal{K}}$ is a pre-additive category.

Functoriality

We want to study the functoriality between the Category of Graphs with the 3-manifolds branched cover $\mathcal{G}_{\mathcal{K}}$ as morphisms and the category of Floer-Kauffman Homology for Graphs with the linear maps a morphisms $\mathcal{C}_{FKh}$. We need to study the existence of a Functor $\mathfrak{F}$ between $\mathcal{G}_{\mathcal{K}}$ and $\mathcal{C}_{FKh} = \mathcal{V}$ where $\mathcal{V}$ is the category of Vector Spaces, such that:

- For $G \in \mathcal{G}_{\mathcal{K}} \xrightarrow{\mathfrak{F}} \widehat{HFG}(G) \in \mathcal{C}_{FKh}$
- Morphisms: Branched cover Spaces $M \xrightarrow{\mathfrak{F}} L$ linear maps
- Composition: For M and $\tilde{M}$ to morphisms in $\mathcal{G}_{\mathcal{K}}$ we want to show

$$\mathfrak{F}(M \circ \tilde{M}) = \mathfrak{F}(M) \circ \mathfrak{F}(\tilde{M})$$

Let

$$G \subset S^3 \xleftarrow{\pi} M \xrightarrow{\pi^{-1}} S^3 \supset G'.$$

We need to find a map between the two Floer Homology groups (M, π, π^{-1})

$$\widehat{HFK}(G, S^3) \xrightarrow{\Upsilon_{(M,\pi,\pi^{-1})}} \widehat{HFK}(G', S^3)$$

(Sketch) We need to use the generators and boundaries explicitly

$$\begin{array}{ccc} \widehat{CFK}_*(G,S^3) & \xrightarrow{\Phi_1} & \widehat{CFK}_*(\pi^{-1}(G),M) \\ \downarrow \partial_* & & \downarrow \partial_* \\ \widehat{CFK}_{*-1}(G,S^3) & \xrightarrow{\Phi_1} & \widehat{CFK}_{*-1}(\pi^{-1}(G),M) \end{array}$$

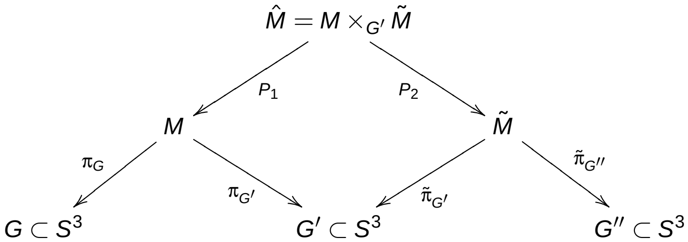
$$\widehat{CFK}_*(G) \xrightarrow{\Psi_2} \widehat{CFK}_*(G \cup G') \xrightarrow{\Psi_3} \widehat{CFK}_*(G')$$

$$\begin{array}{ccc} \widehat{CFK}_*(\pi^{-1}(G'),M) & \xrightarrow{\Phi_4'} & \widehat{CFK}_*(G',S^3) \\ \downarrow \partial_* & & \downarrow \partial_* \\ \widehat{CFK}_{*-1}(\pi^{-1}(G'),M) & \xrightarrow{\Phi_4'} & \widehat{CFK}_{*-1}(G',S^3) \end{array}$$

$$\widehat{HFK}(G,S^3) \xrightarrow{\Upsilon_{(M,\pi,\pi^{-1})}} \widehat{HFK}(G',S^3)$$

where

$$\Upsilon_{(M,\pi,\pi^{-1})} = \Phi_4' \circ \Psi_3 \circ \Psi_2 \circ \Phi_1$$


$\hat{M} = M \times_{G'} \tilde{M}$
P_1
P_2
M
$\tilde{M}$
π_G
$\pi_{G'}$
$\tilde{\pi}_{G'}$
$\tilde{\pi}_{G''}$
$G \subset S^3$
$G' \subset S^3$
$G'' \subset S^3$

$$
\begin{array}{ccc}
\widehat{CFK}_*(G,S^3) & \xrightarrow{\Phi_1} & \widehat{CFK}_*(P_1\circ\pi^{-1}(G), M\times\tilde{M}) \\
\Big\downarrow \partial_* & & \Big\downarrow \partial_* \\
\widehat{CFK}_{*-1}(G,S^3) & \xrightarrow{\Phi_1} & \widehat{CFK}_{*-1}(P_1\circ\pi^{-1}(G), M\times\tilde{M})
\end{array}
$$

$$\widehat{CFK}_*(G) \xrightarrow{\Psi_2} \widehat{CFK}_*(G\cup G') \xrightarrow{\Psi_3} \widehat{CFK}_*(G')$$

$$
\begin{array}{ccc}
\widehat{CFK}_*(P_2\circ\tilde{\pi}_{G''}, M\times\tilde{M}) & \xrightarrow{\Phi'_4} & \widehat{CFK}_*(G'',S^3) \\
\Big\downarrow \partial_* & & \Big\downarrow \partial_* \\
\widehat{CFK}_{*-1}(P_2\circ\tilde{\pi}_{G''}, M\times\tilde{M}) & \xrightarrow{\Phi'_4} & \widehat{CFK}_{*-1}(G'',S^3)
\end{array}
$$

$$\widehat{HFK}(G,S^3) \xrightarrow{\Upsilon_{(M\times\tilde{M},P_1\circ\pi,P_2\circ\tilde{\pi})}} \widehat{HFK}(G',S^3)$$

where

$$\Upsilon_{(M\times\tilde{M},P_1\circ\pi,P_2\circ\tilde{\pi})} = \Phi'_4\circ\Psi_3\circ\Psi_2\circ\Phi_1$$

Consider M in $Hom(G, G')$ specified by a diagram

$$G \subset E \subset S^3 \xleftarrow{\pi_1} M \xrightarrow{\pi_2} S^3 \supset E' \supset G'.$$

We can choose $W = M \times [0,1]$ as a cobordism of M with itself. This has $\partial W = M \cup -M$ with covering maps

$$G \subset S^3 \times [0,1] \xleftarrow{q|_{M\times\{0\}}} W = M \times [0,1] \xrightarrow{q|_{M\times\{1\}}} S^3 \times [0,1] \supset G'$$

branched along the PL surfaces Σ_P in $S^3 \times [0,1]$ with $\partial\Sigma_P = G \cup -G'$. The branched covering maps $q|_{M\times\{0\}} = \pi_1$ and $q|_{M\times\{1\}} = \pi_2$ have the properties that

$$M = q_1^{-1}(S^3 \times \{0\}) = q_1^{-1}(S^3 \times \{1\})$$

$$\begin{array}{ccc}
\pi_1^{-1}(G) \subset M \times \{0\} & \xrightarrow[\Sigma_M]{M\times[0,1]} & M^3 \times \{1\} \supset \pi_2^{-1}(G') \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
G \subset S^3 \times \{0\} & \xrightarrow[\Sigma_P]{S^3\times[0,1]} & S^3 \times \{1\} \supset G'
\end{array}$$

$$\begin{array}{ccc} FKh(\pi_1^{-1}(G)) & \xrightarrow[\Sigma_M]{f_{FKh}} & FKh(\pi_2^{-1}(G')) \\ \downarrow \phi & & \downarrow \phi \\ KKh(G) & \xrightarrow[\Sigma_P]{f_{KKh}} & KKh(G') \end{array}$$

If we use the idea of Kauffman of associating family of links to each graph and use the family of smooth cobordisms to the PL cobordism, and hence we can think by a another functor call it ψ from the link Floer Homology category to the link Khovanov Homology category, which takes the object $\widehat{HFK}((L))$ for a link L to the object $Kh(L)$ and morphism f_{Fh} to the morphism f_{Kh}

$$\begin{array}{ccc} T(\pi_1^{-1}(G)) & \xrightarrow{T(S_\alpha)_{\alpha\in\lambda}} & T(\pi_2^{-1}(G')) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ T(G) & \xrightarrow{T(\Sigma_\alpha)_{\alpha\in\lambda}} & T(G') \end{array}$$

Thank you!