Towards Homotopical Algebraic Quantum Field Theory

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Outline

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3. Discuss our results and inform you about the state-of-the-art of our development of homotopical AQFT

AQFT vs Gauge Theory

AQFT on Lorentzian manifolds (Locally covariant QFT)

◇ **Basic idea:** [Brunetti,Fedenhagen,Verch; . . .]

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⇝ "Coherent assignment of observable algebras to spacetimes"

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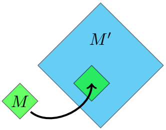
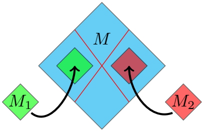
◇ **BFV axioms** (motivated from physics)

Isotony:



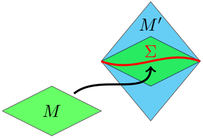
$$\mathfrak{A}(M) \xrightarrow{\text{mono}} \mathfrak{A}(M')$$

Causality:



$$[\mathfrak{A}(M_1), \mathfrak{A}(M_2)] = \{0\}$$

Time-slice:



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Local-to-global property

For every spacetime M, the global algebra $\mathfrak{A}(M)$ can be "recovered" from the algebras $\mathfrak{A}(U)$ corresponding to suitable sub-spacetimes $U \subseteq M$.

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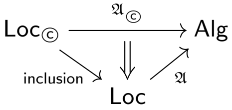
   ✓ $\mathfrak{A}$ determined by restriction $\mathfrak{A}_{©} : \mathsf{Loc}_{©} \to \mathsf{Alg}$ via left Kan extension



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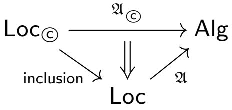
   ✓ true in examples [Lang]



Does $U(1)$-Yang-Mills theory fit into AQFT?

◇ Differential cohomology groups = gauge orbit spaces

$$\widehat{H}^2(M) \cong \frac{\{\text{ principal } U(1)\text{-bundles } P \to M \text{ with connection } A\ \}}{\{\text{ gauge transformations }\}}$$

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$$\mathcal{F}(M) := \{\ h \in \widehat{H}^2(M)\ :\ \delta\,\mathrm{curv}(h) = 0\ \}$$

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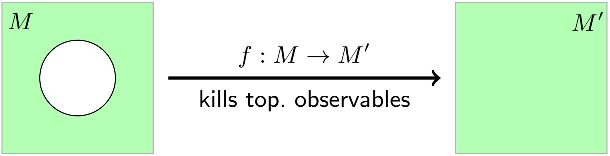
Theorem [Becker,AS,Szabo:1406.1514]

Quantization of smooth Pontryagin dual of $(\mathcal{F}(M), \omega_M)$ defines functor $\mathfrak{A} : \mathsf{Loc} \to \mathsf{Alg}$ which satisfies causality and time-slice, but violates isotony and local-to-global properties.

What is the source of these problems?

◇ Isotony fails because gauge theories carry topological charges



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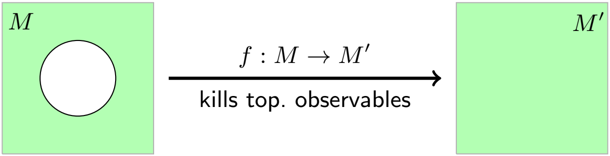

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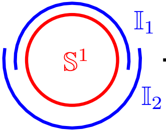
◇ Local-to-global property fails because we took gauge invariant observables

$\widehat{H}^2(\mathbb{S}^1) \cong U(1) \longleftarrow$



$\longrightarrow \widehat{H}^2(\mathbb{I}_{1/2}) \cong 0$

Groupoids vs Gauge Orbit Spaces

Groupoids of gauge fields

- Let's consider for the moment gauge theory on $M \simeq \mathbb{R}^m$
  - Gauge fields $A \in \Omega^1(M, \mathfrak{g})$
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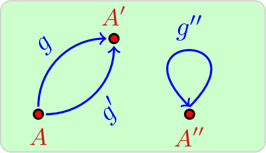

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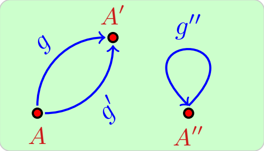

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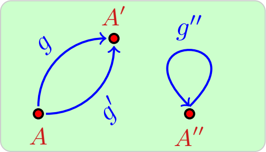

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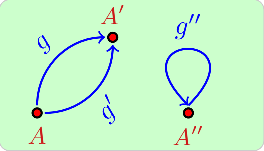

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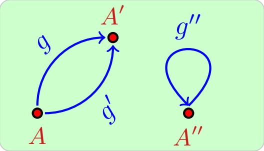

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! Gauge invariant observables ignore the π_1's, hence are incomplete!

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For all manifolds M and all open covers $\{U_\alpha \subseteq M\}$, the canonical map

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NB: Precise formulation of the familiar "gluing up to gauge transformation"

$$\Big\{ \big(\{A_\alpha\},\{g_{\alpha\beta}\}\big) \,:\, A_\beta|_{U_{\alpha\beta}} = A_\alpha|_{U_{\alpha\beta}} \triangleleft g_{\alpha\beta} \,,\;\; g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } U_{\alpha\beta\gamma} \Big\}$$

$$\Updownarrow \; \text{1:1}$$

$$\Big\{ \text{ gauge fields on } M \;\Big\}$$

Cosimplicial observable algebras

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- Taking level-wise smooth functions gives **cosimplicial algebra**

$$\mathcal{O}(M) = \Big(\, C^\infty\big(\Omega^1(M,\mathfrak{g})\big) \;\rightrightarrows\; C^\infty\big(\Omega^1(M,\mathfrak{g}) \times C^\infty(M,G)\big) \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots \,\Big)$$

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⇒ Equivalent description of $\mathcal{O}(M)$ in terms of **differential graded algebra**

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The cosimplicial algebra $\mathcal{O}(M)$ (or equivalently the dg-algebra $\mathcal{O}_{\mathrm{dg}}(M)$) describes non-infinitesimal analogs $C^\infty(M,G)$ of ghost fields $C^\infty(M,\mathfrak{g})$

⇒ BRST formalism for **finite** gauge transformations

Working definition for homotopical AQFT

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$$[\,\cdot\,,\,\cdot\,]_{\mathfrak{A}(M)} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) = \lambda_{f_1,f_2} \circ \mathrm{d} + \mathrm{d} \circ \lambda_{f_1,f_2}$$

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3. *Universality:* $\mathfrak{A} : \mathsf{Loc} \to \mathsf{dgAlg}^{\geq 0}$ is the homotopy left Kan extension of its restriction $\mathfrak{A}_{©} : \mathsf{Loc}_{©} \to \mathsf{dgAlg}^{\geq 0}$.

Working definition (intentionally imprecise)

A **homotopical AQFT** is a (weak) functor $\mathfrak{A} : \mathsf{Loc} \to \mathsf{dgAlg}^{\geq 0}$ to the model category of noncommutative dg-algebras, which satisfies the following axioms:

1. *Causality:* Given causally disjoint $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$, there exist a (coherent) cochain homotopy λ_{f_1,f_2} such that

$$[\,\cdot\,,\,\cdot\,]_{\mathfrak{A}(M)} \circ \big(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)\big) = \lambda_{f_1,f_2} \circ \mathrm{d} + \mathrm{d} \circ \lambda_{f_1,f_2}$$

2. *Time-slice:* Given Cauchy morphism $f : M \to M'$, there exists a (coherent) homotopy inverse $\mathfrak{A}(f)^{-1}$ of $\mathfrak{A}(f)$.
3. *Universality:* $\mathfrak{A} : \mathsf{Loc} \to \mathsf{dgAlg}^{\geq 0}$ is the homotopy left Kan extension of its restriction $\mathfrak{A}_{©} : \mathsf{Loc}_{©} \to \mathsf{dgAlg}^{\geq 0}$.

Precise definition requires **colored operads** [Benini,AS,Woike: work in progress]

homotopical AQFT $:=$ AQFT_∞-algebra + operadic universality

Local-to-global property in Abelian gauge theory

Universal global gauge theory observables

◇ For $G = U(1)$ and $M \simeq \mathbb{R}^m$, $\mathcal{G}(M)$ can be described by chain complex

$$\mathcal{G}_{\mathrm{chain}}(M) = \Big(\ \Omega^1(M) \xleftarrow{\frac{1}{2\pi i}\ \mathrm{d}\log} C^\infty(M, U(1))\ \Big)$$

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Theorem [Benini,AS,Szabo:1503.08839]

1. For $M \simeq \mathbb{R}^m$, $\mathcal{O}_{\copyright}(M)$ and $\mathcal{O}(M)$ are naturally weakly equivalent.
2. For every M, $\mathcal{O}(M)$ weakly equivalent to dual Deligne complex on M.

Toy-models of homotopical AQFT

AQFT on structured spacetimes

◇ **Basic idea:**

1. Consider AQFT $\mathfrak{A} : \mathsf{Str} \to \mathsf{Alg}$ on category of spacetimes with extra geometric structures, i.e. category fibered in groupoids $\pi : \mathsf{Str} \to \mathsf{Loc}$.

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2. Regard $\mathfrak{A}$ as a trivial homotopical AQFT $\mathfrak{A} : \mathsf{Str} \to \mathsf{dgAlg}^{\geq 0}$ via embedding $\mathsf{Alg} \to \mathsf{dgAlg}^{\geq 0}$ of algebras into dg-algebras.

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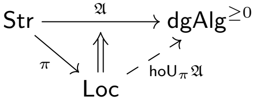

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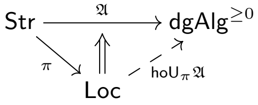

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Theorem [Benini,AS:1610.06071]

Assume that $\pi : \mathsf{Str} \to \mathsf{Loc}$ is strongly Cauchy flabby. Then the homotopy right Kan extension $\mathsf{hoU}_\pi \mathfrak{A} : \mathsf{Loc} \to \mathsf{dgAlg}^{\geq 0}$ satisfies the causality and time-slice axioms of homotopical AQFT. (Coherences just established in low orders.)

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Thanks for your attention.