Towards Homotopical Algebraic Quantum Field Theory

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1. Explain why AQFT is insufficient to describe gauge theories
Outline

1. Explain why

   AQFT is insufficient to describe gauge theories

2. Present ideas/observations indicating that the key to resolve this problem is

   homotopical AQFT := homotopical algebra + AQFT
1. Explain why

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homotopical AQFT := homotopical algebra + AQFT

3. Discuss our results and inform you about the state-of-the-art of our development of homotopical AQFT
AQFT vs Gauge Theory
Basic idea: [Brunetti,Fedenhagen,Verch; ...]

\[
\text{Loc} \overset{\mathcal{A}}{\longrightarrow} \text{Alg}
\]

category of spacetimes \quad category of algebras
AQFT on Lorentzian manifolds (Locally covariant QFT)

◊ **Basic idea**: [Brunetti,Fedenhagen,Verch; ...]

\[ \text{Loc} \xrightarrow{\text{functor } \mathcal{A}} \text{Alg} \]

- category of spacetimes
- category of algebras

⇝ “Coherent assignment of observable algebras to spacetimes”

- \( \mathcal{A}(M) = \) observables we can measure in \( M \)
- \( \mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(M') = \) embedding of observables along \( f : M \to M' \)
**Basic idea:** [Brunetti,Fedenhagen,Verch; ...]

\[
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\[\mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(M') = \text{embedding of observables along } f : M \to M'\]

**BFV axioms** (motivated from physics)

<table>
<thead>
<tr>
<th>Isotony</th>
<th>Causality</th>
<th>Time-slice</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\mathcal{A}(M) \xrightarrow{\text{mono}} \mathcal{A}(M')]</td>
<td>[\mathcal{A}(M_1), \mathcal{A}(M_2)] = {0}]</td>
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</table>
Local-to-global property

For every spacetime \( M \), the global algebra \( \mathcal{A}(M) \) can be “recovered” from the algebras \( \mathcal{A}(U) \) corresponding to suitable sub-spacetimes \( U \subseteq M \).
For every spacetime $M$, the global algebra $\mathfrak{A}(M)$ can be “recovered” from the algebras $\mathfrak{A}(U)$ corresponding to suitable sub-spacetimes $U \subseteq M$.

- Different ways to formalize this property:
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Different ways to formalize this property:

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   - $\mathcal{A}$ determined by restriction $\mathcal{A}_\subset : \text{Loc}_\subset \to \text{Alg}$ via left Kan extension

\[
\begin{tikzcd}
\text{Loc}_\subset & \text{Alg} \\
\text{Loc} \arrow{e}{\mathcal{A}} \arrow{se}{\subset} \arrow{sw}{\mathcal{A}} \arrow{ne}{\text{inclusion}}
\end{tikzcd}
\]
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   ✓ $\mathcal{A}$ determined by restriction $\mathcal{A}_{\subseteq} : \text{Loc}_{\subseteq} \to \text{Alg}$ via left Kan extension

   ✓ true in examples [Lang]
Does $U(1)$-Yang-Mills theory fit into AQFT?

- Differential cohomology groups = gauge orbit spaces

\[
\hat{H}^2(M) \cong \left\{ \text{principal } U(1)\text{-bundles } P \to M \text{ with connection } A \right\}
\left\{ \text{gauge transformations} \right\}
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- Solution spaces of $U(1)$-Yang-Mills theory

$$\mathcal{F}(M) := \left\{ h \in \hat{H}^2(M) : \delta \text{curv}(h) = 0 \right\}$$

are Abelian Fréchet-Lie groups with natural presymplectic structure $\omega_M$. 
Does $U(1)$-Yang-Mills theory fit into AQFT?

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**Theorem** [Becker,AS,Szabo:1406.1514]

Quantization of smooth Pontryagin dual of $(\mathcal{F}(M), \omega_M)$ defines functor $\mathcal{A} : \text{Loc} \to \text{Alg}$ which satisfies **causality** and **time-slice**, but violates **isotony** and **local-to-global properties**.
What is the source of these problems?

- Isotony fails because gauge theories carry topological charges

\[ H^2(M; \mathbb{Z}) \text{ and } H^{m-2}(M; \mathbb{Z}) \]

- Local-to-global property fails because we took gauge invariant observables

\[ \hat{H}^2(S^1) \sim = U(1) \]
[diagram]
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\]

- Local-to-global property fails because we took gauge invariant observables

\[
\hat{H}^2(S^1) \cong U(1) \quad \text{and} \quad \hat{H}^2(\mathbb{I}_{1/2}) \cong 0
\]
Groupoids vs Gauge Orbit Spaces
Groupoids of gauge fields

Let’s consider for the moment gauge theory on $M \simeq \mathbb{R}^m$

- Gauge fields $A \in \Omega^1(M, g)$
- Gauge transformations $g \in C^\infty(M, G)$ acting as $A \triangleleft g = g^{-1} A g + g^{-1} dg$
Groupoids of gauge fields

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- **Groupoid** of gauge fields on $M$

$$G(M) := \Omega^1(M, \mathfrak{g}) \rtimes C^\infty(M, G) =$$
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Two groupoids are “the same” not only when isomorphic, but also when weakly equivalent $\sim$ model category/homotopical algebra
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- Non-redundant information encoded in the groupoid $G(M)$
  1. Gauge orbit space $\pi_0(G(M)) = \Omega^1(M, g)/C^\infty(M, G)$
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$$G(M) := \Omega^1(M, g) \rtimes C^\infty(M, G) = \begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) [circle,fill,inner sep=2pt] {}; 
  \node (A') at (1,0) [circle,fill,inner sep=2pt] {}; 
  \node (A'') at (2,0) [circle,fill,inner sep=2pt] {}; 
  \draw [->] (A) edge node [left] {$g$} (A'); 
  \draw [->] (A') edge node [right] {$g''$} (A''); 
\end{tikzpicture}
\end{array}$$

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! Gauge invariant observables ignore the $\pi_1$’s, hence are incomplete!
Groupoids and local-to-global properties

Groupoids of gauge fields satisfy homotopical local-to-global property
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Groupoids of gauge fields satisfy homotopical local-to-global property

**Homotopy sheaf property**

For all manifolds $M$ and all open covers $\{U_{\alpha} \subseteq M\}$, the canonical map

$$G(M) \longrightarrow \text{holim} \left( \prod_{\alpha} G(U_{\alpha}) \Longrightarrow \prod_{\alpha\beta} G(U_{\alpha\beta}) \Longrightarrow \prod_{\alpha\beta\gamma} G(U_{\alpha\beta\gamma}) \Longrightarrow \cdots \right)$$

is a weak equivalence in Grpd.
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Groupoids of gauge fields satisfy homotopical local-to-global property

Homotopy sheaf property

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$$

is a weak equivalence in Grpd.

**NB:** Precise formulation of the familiar “gluing up to gauge transformation”

$$
\left\{ (\{A_\alpha\}, \{g_{\alpha\beta}\}) : A_\beta|_{U_{\alpha\beta}} = A_\alpha|_{U_{\alpha\beta}} \triangleleft g_{\alpha\beta} , \ g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \text{ on } U_{\alpha\beta\gamma} \right\}
$$

$$
\Leftrightarrow 1:1
$$

$$
\left\{ \text{gauge fields on } M \right\}
$$
Cosimplicial observable algebras
What are “function algebras” on groupoids?

- QFT needs quantized ‘algebras’ of functions on the ‘spaces’ of fields
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  - Space of fields \( \mathcal{F}(M) \) is set (+ smooth structure)
  - \( O(M) = C^\infty(\mathcal{F}(M)) \) has the structure of an algebra
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- Nerve construction $N : \text{Grpd} \to \text{sSet}$ assigns the simplicial set
  
  $$N(\mathcal{G}(M)) = \left( \Omega^1(M, \mathfrak{g}) \leftarrow \Omega^1(M, \mathfrak{g}) \times C^\infty(M, G) \leftarrow \cdots \right)$$
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- Taking level-wise smooth functions gives cosimplicial algebra
  $$O(M) = \left( C^\infty(\Omega^1(M, \mathfrak{g})) \rightleftharpoons C^\infty(\Omega^1(M, \mathfrak{g}) \times C^\infty(M, G)) \rightleftharpoons \cdots \rightleftharpoons \cdots)$$
Relation to the BRST formalism and ghost fields

- Dual Dold-Kan correspondence gives equivalence $\text{cAlg} \Leftrightarrow \text{dgAlg}^{\geq 0}$

- Considering only infinitesimal gauge transformations $C_\infty(M, g)$

  - Van Est map $\rightarrow C_\infty(\Omega^1(M, g))$

  - Gauge field observables $\otimes \wedge$ $\cdot$ $C_\infty(M, g)$

  - Ghost field observables $\cdot$ $\star$ $C_\infty(M, g)$

- The cosimplicial algebra $O(M)$ (or equivalently the dg-algebra $O_{dg}(M)$) describes non-infinitesimal analogs $C_\infty(M, G)$ of ghost fields $C_\infty(M, g)$

  - $\Rightarrow$ BRST formalism for finite gauge transformations
Relation to the BRST formalism and ghost fields

- Dual Dold-Kan correspondence gives equivalence $\text{cAlg} \leftrightarrow \text{dgAlg}_{\geq 0}$

$\implies$ Equivalent description of $\mathcal{O}(M)$ in terms of **differential graded algebra**

$$\mathcal{O}_{\text{dg}}(M) = \left( C^{\infty}(\Omega^1(M, \mathfrak{g})) \longrightarrow C^{\infty}(\Omega^1(M, \mathfrak{g}) \times C^{\infty}(M, G)) \longrightarrow \cdots \right)$$
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○ Dual Dold-Kan correspondence gives equivalence \( \text{cAlg} \iff \text{dgAlg}^{\geq 0} \)

\( \Rightarrow \) Equivalent description of \( \mathcal{O}(M) \) in terms of \textbf{differential graded algebra}

\[
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\]

○ Considering only infinitesimal gauge transformations \( C^\infty(M, \mathfrak{g}) \)

\[
\mathcal{O}_{\text{dg}}(M) \xrightarrow{\text{van Est map}} \underbrace{C^\infty\left( \Omega^1(M, \mathfrak{g}) \right)}_{\text{gauge field observables}} \otimes \underbrace{\wedge^\bullet C^\infty(M, \mathfrak{g})^*}_{\text{ghost field observables}}
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Relation to the BRST formalism and ghost fields

○ Dual Dold-Kan correspondence gives equivalence $\text{cAlg} \cong \text{dgAlg}_{\geq 0}$

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$\Rightarrow$ BRST formalism for finite gauge transformations
Working definition for homotopical AQFT
A **homotopical AQFT** is a (weak) functor $\mathcal{A} : \text{Loc} \to \text{dgAlg}_{\geq 0}$ to the model category of noncommutative dg-algebras, which satisfies the following axioms:
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1. **Causality:** Given causally disjoint $M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$, there exist a (coherent) cochain homotopy $\lambda_{f_1,f_2}$ such that

$$\left[ \cdot, \cdot \right]_{\mathcal{A}(M)} \circ (\mathcal{A}(f_1) \otimes \mathcal{A}(f_2)) = \lambda_{f_1,f_2} \circ d + d \circ \lambda_{f_1,f_2}$$

2. **Time-slice:** Given Cauchy morphism $f : M \to M'$, there exists a (coherent) homotopy inverse $\mathcal{A}(f)^{-1}$ of $\mathcal{A}(f)$.

3. **Universality:** $\mathcal{A} : \text{Loc} \to \text{dgAlg}^{\geq 0}$ is the homotopy left Kan extension of its restriction $\mathcal{A}^c : \text{Loc}^c \to \text{dgAlg}^{\geq 0}$.

Precise definition requires colored operads [Benini, AS, Woike: work in progress].
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homotopical AQFT $:= \text{AQFT}_\infty$-algebra + operadic universality
Local-to-global property in Abelian gauge theory
Universal global gauge theory observables

- For $G = U(1)$ and $M \simeq \mathbb{R}^m$, $\mathcal{G}(M)$ can be described by chain complex

$$\mathcal{G}_{\text{chain}}(M) = \left( \Omega^1(M) \leftarrow \frac{1}{2\pi i} d \log \rightarrow C^\infty(M, U(1)) \right)$$
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$$G_{\text{chain}}(M) = \left( \Omega^1(M) \xleftarrow{\frac{1}{2\pi i}} d \log \xrightarrow{} C^\infty(M, U(1)) \right)$$

- Smooth Pontryagin dual cochain complex of observables

$$O \hat{\otimes}(M) := \left( \Omega^{m-1}_c(M) \xrightarrow{d} \Omega^m_{c;\mathbb{Z}}(M) \right)$$
Universal global gauge theory observables

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\]

- Smooth Pontryagin dual cochain complex of observables

\[
\mathcal{O}_{\text{c}}(M) := \left( \Omega^{m-1}_c(M) \xrightarrow{d} \Omega^m_c(M) \right)
\]

- Homotopy left Kan extension of $\mathcal{O}_{\text{c}} : \text{Loc}_{\text{c}} \to \text{Ch}^{\geq 0}$

\[
\mathcal{O}(M) := \text{hocolim} \left( \mathcal{O}_{\text{c}} : \text{Loc}_{\text{c}} \downarrow M \to \text{Ch}^{\geq 0} \right)
\]
Universal global gauge theory observables

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$$\mathcal{G}_{\text{chain}}(M) = \left( \Omega^1(M) \xleftarrow{\frac{1}{2\pi i}} \log \right) \mathcal{C}^\infty(M, U(1))$$

- Smooth Pontryagin dual cochain complex of observables

$$\mathcal{O}_\odot(M) := \left( \Omega_{\text{c}}^{m-1}(M) \xrightarrow{\text{d}} \Omega_{\text{c};\mathbb{Z}}^m(M) \right)$$

- Homotopy left Kan extension of $\mathcal{O}_\odot : \text{Loc}\odot \to \text{Ch}^{\geq 0}$

$$\mathcal{O}(M) := \text{hocolim} \left( \mathcal{O}_\odot : \text{Loc}\odot \downarrow M \to \text{Ch}^{\geq 0} \right)$$

**Theorem** [Benini,AS,Szabo:1503.08839]

1. For $M \simeq \mathbb{R}^m$, $\mathcal{O}_\odot(M)$ and $\mathcal{O}(M)$ are naturally weakly equivalent.

2. For every $M$, $\mathcal{O}(M)$ weakly equivalent to dual Deligne complex on $M$. 
Toy-models of homotopical AQFT
AQFT on structured spacetimes

◊ Basic idea:

1. Consider AQFT $\mathcal{A} : \text{Str} \to \text{Alg}$ on category of spacetimes with extra geometric structures, i.e. category fibered in groupoids $\pi : \text{Str} \to \text{Loc}$. ($\pi^{-1}(M)$ is groupoid of structures over $M$, e.g. spin structures, gauge fields)
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2. Regard $\mathcal{A}$ as a trivial homotopical AQFT $\mathcal{A} : \text{Str} \rightarrow \text{dgAlg}^{\geq 0}$ via embedding $\text{Alg} \rightarrow \text{dgAlg}^{\geq 0}$ of algebras into dg-algebras.
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3. Perform homotopy right Kan extension

\[
\text{Str} \xrightarrow{\mathcal{A}} \text{dgAlg} \geq 0 \xleftarrow{\pi} \text{Loc}
\]

\[
\xrightarrow{\text{hoU}_{\pi} \mathcal{A}}
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to induce a nontrivial homotopical AQFT $\text{hoU}_{\pi} \mathcal{A}$ on Loc.
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\[
\begin{array}{c}
\text{Str} \\
\downarrow \pi \\
\text{Loc} \\
\uparrow \mathcal{A} \\
\text{dgAlg}_{\geq 0} \\
\uparrow \text{hoU}_\pi \mathcal{A} \\
\end{array}
\]


to induce a *nontrivial* homotopical AQFT $\text{hoU}_\pi \mathcal{A}$ on $\text{Loc}$.

**Theorem** [Benini,AS:1610.06071]

Assume that $\pi : \text{Str} \to \text{Loc}$ is strongly Cauchy flabby. Then the homotopy right Kan extension $\text{hoU}_\pi \mathcal{A} : \text{Loc} \to \text{dgAlg}_{\geq 0}$ satisfies the *causality and time-slice axioms* of homotopical AQFT. (Coherences just established in low orders.)
Summary and Outlook
Quantum gauge theories are **NOT** contained in the AQFT framework.

- Already very promising results:
  - Local-to-global property of observables \[\text{[Benini,AS,Szabo:1503.08839]}\]
  - Toy-models of homotopical AQFT \[\text{[Benini,AS:1610.06071]}\]
  - Yang-Mills stack and stacky Cauchy problem \[\text{[Benini,AS,Schreiber:1704.01378]}\]

- Open problems/Work in progress:
  1. Develop operadic approach to homotopical AQFT to control coherences \[\text{[Benini,AS,Woike: work in progress]}\]
  2. Construct proper examples of dynamical and quantized gauge theories \[\text{[Benini,AS,Szabo: work in progress]}\]
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To capture crucial homotopical features of classical gauge theories, one needs “higher algebras” to formalize quantum gauge theories.
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Thanks for your attention.