Segal-type models of higher categories

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Two motivating examples

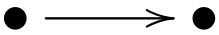
Two prototype examples in dimension 2:

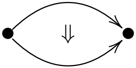
a) The 2-dimensional structure with
Objects = categories
1-morphisms = functors
2-morphisms = natural transformations.

b) The 2-dimensional structure with
Objects = points of a space X
1-morphisms = paths in X
2-morphisms = 2-tracks (equivalence classes of homotopies between paths).

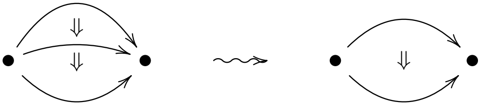
- Objects are also called 0-cells and k-morphisms are called k-cells.
- In both examples, we can use the pictorial representation

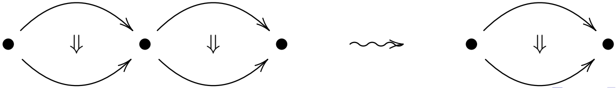
Objects 

1-morphisms 

2-morphisms 

Vertical and horizontal compositions





Two motivating examples, cont.

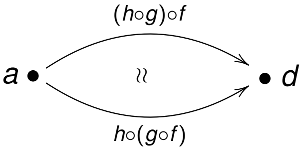
Main difference between examples a) and b):

a) All compositions are associative and unital. This is a strict 2-category.

b) Composition of paths is associative and unital only up to homotopy; given paths

$$\underset{a}{\bullet} \xrightarrow{f} \underset{b}{\bullet} \xrightarrow{g} \underset{c}{\bullet} \xrightarrow{h} \underset{d}{\bullet}$$

there is a homotopy



The structure we obtain is a weak 2-category.

Idea of strict n-category: in a strict n-category there are cells in dimension $0, \dots, n$, identity cells and compositions which are associative and unital. Each k-cell has source and target which are $(k-1)$-cells, $1 \leq k \leq n$.

Strict n-categories are defined by iterated enrichment:

$$1\text{-}Cat = Cat, \qquad n\text{-}Cat = ((n-1)\text{-}Cat)\text{-}Cat$$

When all cells have inverses, we obtain a strict n-groupoid

Strict *n*-groupoids and *n*-types

- An *n*-type is a topological space whose homotopy groups vanish in dimension higher than *n*.
- *n*-types are the building blocks of spaces via the Postnikov decomposition.
- Fact: Strict *n*-groupoids do not model *n*-types when $n > 2$.

This was one of the motivations for the development of weak *n*-categories: in the weak *n*-groupoid case it gives an algebraic model of *n*-types (homotopy hypothesis).

Idea of weak n-category: in a weak n-category there are cells in dimension $0, \dots, n$, identity cells and compositions which are associative and unital up to an invertible cell in the next dimension, in a coherent way.

- In dimensions $n = 2, 3$ it is possible to give an explicit definition of the axioms with the notions of bicategory and tricategory.
- For general n there are several different models of weak n-categories and weak n-groupoids.

Internal categories and internal groupoids

Definition

- An *internal category* in a category $\mathcal{C}$ with pullbacks consists of a diagram in $\mathcal{C}$

$$C_1 \times_{C_0} C_1 \xrightarrow{c} C_1 \underset{s}{\overset{d_0, d_1}{\rightleftarrows}} C_0$$

  where these maps satisfies the axiom of a category.

- An *internal groupoid* in $\mathcal{C}$ is an internal category with all morphisms invertible.

- Denote by Cat $\mathcal{C}$ the category of internal categories and internal functors.

Definition

n-fold categories are defined inductively as

$$\mathrm{Cat}^1 = \mathrm{Cat}$$

$$\mathrm{Cat}^n = \mathrm{Cat}(\mathrm{Cat}^{n-1})$$

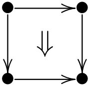
Example: double categories

- Let $X \in \mathrm{Cat}(\mathrm{Cat})$

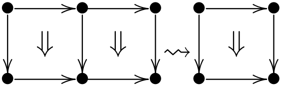
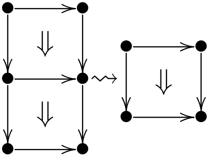
  $X_0 \in \mathrm{Cat}$ has

  objects • morphisms 

  $X_1 \in \mathrm{Cat}$ has

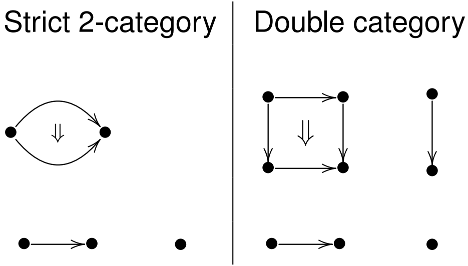
  objects  morphisms 

  Thus squares can be composed horizontally and vertically

  All compositions are associative and unital.

Strict 2-categories versus double categories



- Note: the picture on the right becomes the one on the left when all vertical morphisms are identities.

Strict n-categories versus n-fold categories

- There is an embedding

$$n\text{-Cat} \hookrightarrow \mathrm{Cat}^n.$$

  A strict n-category $X \in n$-Cat is a n-fold category in which certain substructures are discrete (that is just sets).

- This discreteness condition is called the globularity condition.

- The sets underlying these discrete substructures are the sets of cells in the strict n-category.

The category n-Cat is too small to model weak n-category while Cat^n is too large. Is there an intermediate category

$$n\text{-Cat} \hookrightarrow \, ? \, \hookrightarrow \mathrm{Cat}^n$$

which is a model of weak n-categories?

The answer is provided by the category $\mathrm{Cat}^n_{\mathrm{wg}}$ of weakly globular n-fold categories.

Some historical development

- A pioneering work on the use of n-fold structures in connection with homotopy theory is Loday's Cat^n-groups as a model of path-connected $(n+1)$-types.

- This was also investigated by Bullejos-Cegarra-Duskin with a different approach, and led Brown to a proof a higher order Van-Kampen theorem with interesting computational applications.

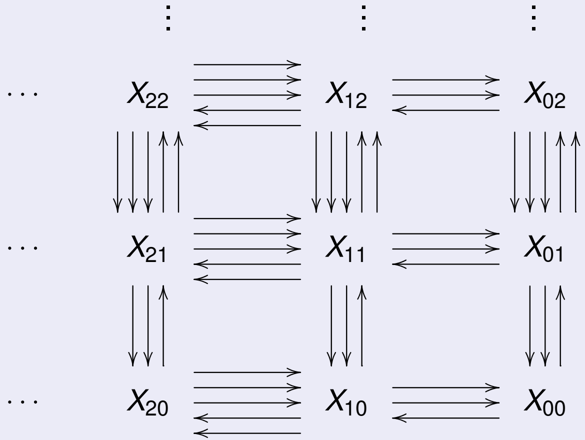
- A combinatorially different model was also given by Porter and by Ellis-Steiner in terms of crossed n-cubes.

- The notion of weak globularity first arose in a special case in relating Loday's model to the Tamsamani-Simpson model in the path-connected case ([P., Adv. Math. 2009]).

Simplicial combinatorics.

- Let Δ be the simplicial category. Its objects are finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$ and its morphisms are non decreasing monotone functions.
- The functor category $[\Delta^{op}, \mathcal{C}]$ is the category of simplicial objects and simplicial maps in $\mathcal{C}$.
- Let $\Delta^{n^{op}} = \Delta^{op} \times \overset{n}{\cdots} \times \Delta^{op}$.
- Multi-simplicial objects in $\mathcal{C}$ are functors $[\Delta^{n^{op}}, \mathcal{C}]$.

Example: Bisimplicial object



- There is a fully faithful nerve functor

$$N : \mathrm{Cat}\, \mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\, \mathcal{C}$

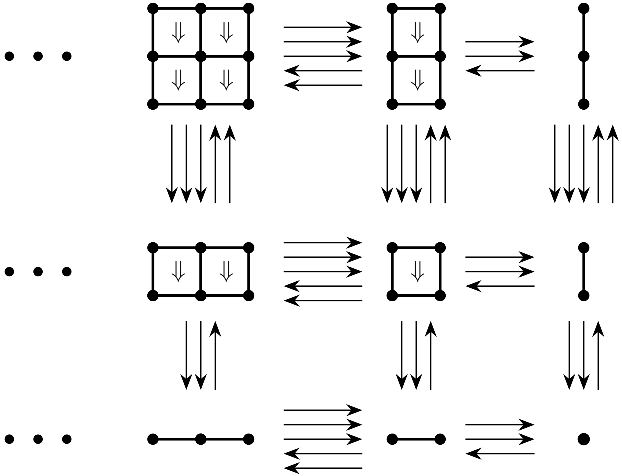
$$NX \qquad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X_1 \times_{X_0} X_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow}\; X_1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow}\; X_0$$

- By iterating the nerve construction, we obtain fully faithful multinerve functors

$N_{(n)} : \mathrm{Cat}^{\mathrm{n}} \to [\Delta^{n^{op}}, \mathrm{Set}]$

$J_n : \mathrm{Cat}^{\mathrm{n}} \to [\Delta^{n-1^{op}}, \mathrm{Cat}]$

Example: the double nerve of a double category



Higher categories via multi-simplicial objects.

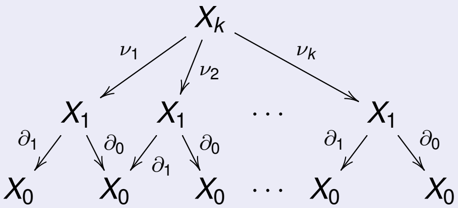
Multi-simplicial objects are a good environment for the definition of higher categorical structures because there are natural candidates for the compositions given by the Segal maps.

Our structures are based on $[\Delta^{{n-1}^{op}}, \mathrm{Cat}]$. These can be used to model higher categories by imposing additional conditions to encode:

i) The sets of cells in dimension 0 up to n.
ii) The behavior of the compositions.
iii) The higher categorical equivalences.

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks.
Denote $X[k] = X_k$.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j-1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

- Recall the nerve functor

$$N : \mathrm{Cat}\, \mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\, \mathcal{C}$

$$NX \qquad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} X_1 \times_{X_0} X_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} X_1 \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} X_0$$

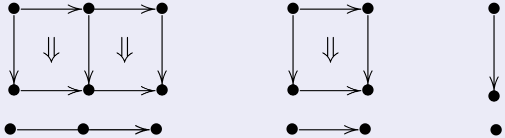
Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps $X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ are isomorphisms.

Segal maps and multi-simplicial objects.

For each $X \in [\Delta^{n^{op}}, \mathcal{C}]$ we have Segal maps in each of the n simplicial directions.

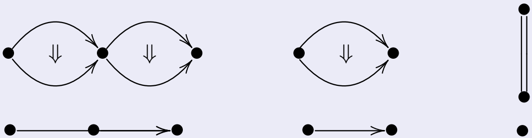
Using these Segal maps one can characterize the image of the multinerve $J_n : \mathrm{Cat}^{\mathrm{n}} \to [\Delta^{n^{op}}, \mathrm{Cat}]$ and $J_n : n\text{-}\mathrm{Cat} \to [\Delta^{n^{op}}, \mathrm{Cat}]$

- Double category $X \in \mathrm{Cat}(\mathrm{Cat}) \in [\Delta^{op}, \mathrm{Cat}]$



$$\cdots \quad X_1 \times_{X_0} X_1 \rightrightarrows X_1 \rightleftarrows X_0$$

- Strict 2-category $X \in 2\text{-}\mathrm{Cat} \in [\Delta^{op}, \mathrm{Cat}]$



$$\cdots \quad X_1 \times_{X_0} X_1 \rightrightarrows X_1 \rightleftarrows X_0$$

Segal-type models.

- We discuss three Segal-type models of weak *n*-categories, collectively denoted Seg^n

$$\mathsf{Ta}^n \hookrightarrow \mathsf{Ta}^n_{wg} \hookleftarrow \mathsf{Cat}^n_{wg}$$

- We have $\mathsf{Seg}^n \subset [\Delta^{n-1^{op}}, \mathsf{Cat}]$.

Building on dimensions.

- Recall in a weak n-category we want to have k-cells with source and target being $(k-1)$-cells for $1 \leq k \leq n$.
- $\mathrm{Seg}^{\mathrm{n}}$ is built by induction on n starting with $\mathrm{Seg}^1 = \mathrm{Cat}$.
- For each $n > 1$:

$$\mathrm{Seg}^{\mathrm{n}} \hookrightarrow [\Delta^{op}, \mathrm{Seg}^{\mathrm{n-1}}]$$

We encode in two ways the sets of cells of $X \in \mathrm{Seg}^{\mathrm{n}}$

i) Globularity condition:

$X_0, \quad X_{1...10-}^{\;r} \qquad 1 \leq r < n-1 \quad$ discrete

ii) Weak globularity condition:

$X_0, \quad X_{1...10-}^{\;r} \qquad 1 \leq r < n-1 \quad$ homotopically discrete

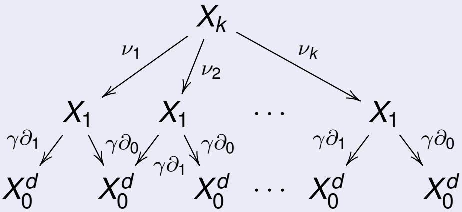
Let $X \in \mathrm{Seg}^{\mathrm{n}} \subset [\Delta^{op}, \mathrm{Seg}^{\mathrm{n}-1}]$ to be such that X_0 satisfies i) or ii).
There is also a discretization map $\gamma : X_0 \to X_0^d$ where X_0^d is discrete.

The n^{th} truncation functor.

- There is a functor
$$p^{(n)} : \mathrm{Seg^n} \to \mathrm{Seg^{n-1}}$$
which divides out by the highest dimensional invertible cells.

- This is used to define the notion of n-equivalence.

Given $X \in \mathrm{Seg}^n \subset [\Delta^{op}, \mathrm{Seg}^{n-1}]$, consider the commuting diagram



where $k \geq 2$, $\nu_j = X(r_j)$, $r_j(0) = j-1$, $r_j(1) = j$. This gives the induced Segal map

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

The induced Segal maps condition.

To define $X \in \mathrm{Seg}^{\mathrm{n}} \subset [\Delta^{op}, \mathrm{Seg}^{\mathrm{n}-1}]$ we require the induced Segal maps

$$X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

to be $(n-1)$-equivalences.

This condition controls the behaviour of the compositions of higher cells.

Summary of main common features of $\mathrm{Seg}^{\mathrm{n}}$.

- Inductive multi-simplicial definition.
- Globularity/weak globularity condition.
- Functor $p^{(n)} : \mathrm{Seg}^{\mathrm{n}} \to \mathrm{Seg}^{\mathrm{n-1}}$ and n-equivalences.
- $(n-1)$-equivalences of the induced Segal maps

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

The three models.

Three different models corresponding to different behavior of:

Induced Segal maps $\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$

Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$

	X_0	$\hat{\mu}_k$	η_k
Ta^n	discrete	$(n-1)$-eq	$(n-1)$-eq
Cat^n_{wg}	homotopically discrete	$(n-1)$-eq	isomorphisms
Ta^n_{wg}	homotopically discrete	$(n-1)$-eq	-

Model comparison results.

Theorem (P. case n>2; P. and Pronk case n=2)

There are functors

$Q_n : \mathrm{Ta^n} \to \mathrm{Cat^n_{wg}}$ *rigidification functor*

$Disc_n : \mathrm{Cat^n_{wg}} \to \mathrm{Ta^n}$ *discretization functor*

producing n-equivalent objects in $\mathrm{Ta^n_{wg}}$.

Corollary

There is an equivalence of categories

$$\mathrm{Ta^n}/\sim^n \simeq \mathrm{Cat^n_{wg}}/\sim^n$$

The homotopy hypothesis.

From the comparison theorem between $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ and Ta^{n} we obtain

Theorem

There is a subcategory $\mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}} \subset \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ of groupoidal weakly globular n-fold categories such that there is an equivalence of categories

$$\mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}}/\sim^n \simeq Ho(n\text{-}types)\ .$$

There is an explicit description of the functor n-types $\rightarrow \mathrm{GCat}^{\mathrm{n}}_{\mathrm{wg}}$ using a construction of [Blanc and P., Alg.Geom. Topol. 2015].

Research Monograph:

S.Paoli, Segal-type models of higher categories, 2017, (310 pages) available at arXiv.1707.01868.

Thank you for your attention