Homotopy algebras versus algebras up to homotopy

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Joint work with Oriol Raventós and Andrew Tonks

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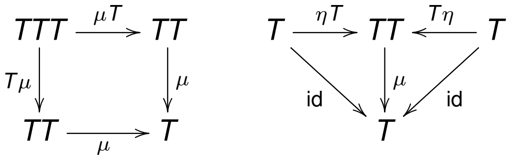
Q is a cofibrant replacement functor on $\mathcal{M}$

Monads

A **monad** on a category $\mathcal{C}$ is a triple (T, η, μ) where $T\colon \mathcal{C} \to \mathcal{C}$ is a functor and $\eta\colon \mathrm{Id} \to T$ and $\mu\colon TT \to T$ are natural transformations such that the following diagrams commute:



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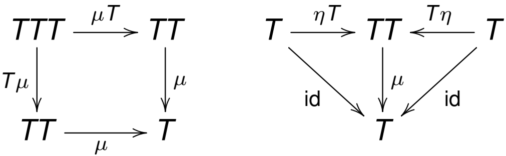


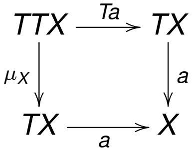
For every pair of adjoint functors

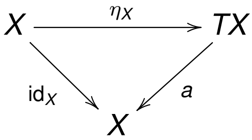
$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

with unit η and counit ε, the triple $(GF, \eta, G\varepsilon F)$ is a monad.

Algebras over monads

If (T, η, μ) is a monad on a category $\mathcal{C}$, then a ***T*-algebra** is a pair (X, a) with $a\colon TX \to X$ such that the following diagrams commute:





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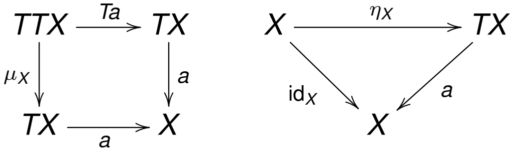

A morphism of T-algebras $(X, a) \to (Y, b)$ is an arrow $\varphi\colon X \to Y$ in $\mathcal{C}$ such that $\varphi \circ a = b \circ T\varphi$.

Eilenberg–Moore adjunction

Thus, the T-algebras on $\mathcal{C}$ form a category, denoted by $\boldsymbol{\mathcal{C}^T}$ and called the **Eilenberg–Moore category** of T.

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This adjunction is terminal among all adjunctions whose associated monad is T.

Monads on model categories

Let $\mathcal{M}$ be a **model category** and let T be a monad on $\mathcal{M}$. Consider the Eilenberg–Moore adjunction

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A model structure on $\mathcal{M}^T$ is called **transferred** if U creates weak equivalences and fibrations.

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- Sufficient conditions for the existence of transferred model structures are known when $\mathcal{M}$ is cofibrantly generated: [Crans, 1995]; [Schwede–Shipley, 2000]; [Berger–Moerdijk, 2003]; [Johnson–Noel, 2014]; [Batanin–Berger, 2017].

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- We will display an example where such a transferred model structure does *not* exist.

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If $\mathcal{M}^T$ admits a transferred model structure, then the forgetful functor U preserves fibrations and trivial fibrations. Hence

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$$FQ : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}^T) : U$$

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This is **not** equivalent in general to the Eilenberg–Moore adjunction

$$\mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M})^{TQ}$$

of the derived monad TQ.

Examples

Let Spec be a monoidal model category of spectra. Let E be a ring spectrum. Let $\boldsymbol{TX = E \wedge X}$**.** Then T-algebras are E-module spectra.

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Hence $\mathrm{Ho(Spec)}^{TQ}$ is not equivalent to $\mathrm{Ho(Spec}^{T})$ if $E = HR$ and R has global dimension bigger than 1, since

$$\mathbf{Ho(Spec^{\boldsymbol{T}})}(HA, \Sigma^k HB) \cong \mathcal{D}(R)(HA, \Sigma^k HB) \cong \mathrm{Ext}^k_R(A, B)$$

for all R-modules A and B and $k \geq 0$, while

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Note that for $R \subseteq \mathbb{Q}$ the two categories *are* equivalent.

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- For an **E_∞-operad,** the objects of $\mathrm{Ho}(\mathrm{sSet}^T)$ are E_∞-spaces, while those of $\mathrm{Ho}(\mathrm{sSet})^T$ are H_∞-spaces.

Homotopical localizations

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Examples:

- Homological localizations
- Localizations at primes
- v_n-localizations
- Postnikov sections

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Examples:

- Cellular approximations
- Connective covers

Earlier results

[C–Gutiérrez–Moerdijk–Vogt, 2010]:

Let E be a ring spectrum. Let X be an E-module. Let L be any exact f-localization in a monoidal model category of spectra. Then LX acquires an E-module structure such that the localization map $\ell_X \colon X \to LX$ is an E-module map, and this E-module structure on LX is unique up to homotopy.

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- **Corollary:** If X splits as a wedge of Eilenberg–Mac Lane spectra then so does LX.
- **Question:** Does L induce a localization on E-modules?

Main results

[C–Raventós–Tonks, 2017]:

Theorem 1 *For a monad T on a model category $\mathcal{M}$, suppose that $\mathcal{M}^T$ has a transferred model structure and let Q denote a cofibrant replacement functor on $\mathcal{M}$. Let f be a map in $\mathcal{M}$ such that $\mathcal{M}_f$ exists. Then L_f lifts to* $\mathbf{Ho}(\mathcal{M})^{TQ}$ *if and only if T sends f-equivalences between cofibrant objects to f-equivalences.*

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- The statement that L_f **lifts** to $\mathrm{Ho}(\mathcal{M})^{TQ}$ means that there is a localization functor L on $\mathrm{Ho}(\mathcal{M})^{TQ}$ such that $UL = L_f U$.
- An ***f*-equivalence** is a map $X \to Y$ in $\mathcal{M}$ such that the induced map $L_f X \to L_f Y$ is a weak equivalence.

Main results

Theorem 2 *For a monad T on a model category $\mathcal{M}$, suppose that $\mathcal{M}^T$ has a transferred model structure and let*

$$F : \mathcal{M} \rightleftarrows \mathcal{M}^T : U$$

be the Eilenberg–Moore factorization of T. Let f be a map in $\mathcal{M}$ between cofibrant objects such that the left Bousfield localizations $\mathcal{M}_f$ and $(\mathcal{M}^T)_{Ff}$ exist. Then the following statements are equivalent:

(i) *L_f lifts to* **$\mathrm{Ho}(\mathcal{M}^T)$.**

(ii) *The functor U sends Ff-equivalences to f-equivalences.*

(iii) *L_{Ff} is a lift of L_f to* $\mathrm{Ho}(\mathcal{M}^T)$.

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(iii) *L_{Ff} is a lift of L_f to $\mathrm{Ho}(\mathcal{M}^T)$.*

Note: If L_f lifts to $\mathrm{Ho}(\mathcal{M}^T)$ then it also lifts to $\mathrm{Ho}(\mathcal{M})^{TQ}$.

Back to module spectra

Corollary *Let E be a cofibrant spectrum and let $f\colon A \to B$ be a map between cofibrant spectra. Suppose that E is a ring spectrum, and let U denote the forgetful functor from E-modules to spectra. If L_f is exact then L_f lifts to E-modules.*

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Proof. The forgetful functor has a Quillen *right* adjoint, namely $\mathrm{Map}(E, -)$. This implies that U sends Ff-equivalences between cofibrant objects to Tf-equivalences (recall that $T = UF$). Since $Tf = E \wedge f$, we need to prove that $E \wedge f$ is an f-equivalence in order to use Theorem 2. This follows from a standard argument:

$$\mathrm{Map}(E \wedge B, X) \simeq \mathrm{Map}(E, \mathrm{Map}(B, X)) \simeq \mathrm{Map}(E, \mathrm{Map}(A, X)) \simeq \mathrm{Map}(E \wedge A, X)$$

for every f-local spectrum X, since L_f is exact. $\square$

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Choose $\boldsymbol{E = K(n)}$**,** the n-th Morava K-theory at a prime p, and $\boldsymbol{f\colon S \to 0}$**,** where S denotes the sphere spectrum. Thus $L_f X = P_{-1} X$, a Postnikov section at level -1.

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Then $P_{-1}K(n)$ is the homotopy fibre of the connective cover $k(n) \to K(n)$. Since $H\mathbb{Z}/p \wedge K(n) = 0$ but $H\mathbb{Z}/p \wedge k(n) \neq 0$, we infer that $H\mathbb{Z}/p \wedge P_{-1}K(n) \neq 0$. Hence $P_{-1}K(n)$ is not a homotopy retract of $K(n) \wedge P_{-1}K(n)$ and this implies that $P_{-1}K(n)$ is not a $K(n)$-module.

In conclusion, $\boldsymbol{P_{-1}}$ **does not preserve** $\boldsymbol{K(n)}$**-modules.**

Another counterexample

Let $\boldsymbol{TX = K(n) \wedge X}$ on spectra and choose $f\colon S \to 0$ as above. Look at the adjunctions

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Assume that $(\mathrm{Spec}_f)^T$ admits a transferred model structure. Then the model categories $(\mathrm{Spec}_f)^T$ and $(\mathrm{Spec}^T)_{Ff}$ have the same trivial fibrations and the same fibrant objects, from which it follows that they have the same weak equivalences. Therefore U sends Ff-equivalences to f-equivalences and Theorem 2 implies that L_f lifts to $\mathrm{Ho}(\mathrm{Spec}^T)$. But this cannot be true, since we just saw that L_f does not preserve $K(n)$-modules.

Hence $(\mathrm{Spec}_f)^T$ has no transferred model structure.

Further results

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Our results have obvious analogues for cellularizations.