Multi derivation Maurer-Cartan algebras and sh-Lie-Rinehart algebras

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Abstract

Given a commutative algebra A over a ground ring R and an A-module L, a Maurer-Cartan algebra relative to A and L is the graded A-algebra $\mathrm{Alt}_A(L, A)$ of A-valued A-multilinear alternating froms on L together with an R-derivation d that turns $(\mathrm{Alt}_A(L, A), d)$ into a differential graded R-algebra. An example of a Maurer-Cartan is the de Rham algebra of a smooth manifold; another example is the familiar differential graded algebra of alternating forms on a Lie algebra $\mathfrak{g}$ with values in the ground field, endowed with the standard Lie algebra cohomology operator.

Abstract continued

We extend the classical characterization of a finite-dimensional Lie algebra $\mathfrak{g}$ in terms of its Maurer-Cartan algebra to sh Lie-Rinehart algebras. To this end, we first develop a characterization of sh Lie-Rinehart algebras in terms of differential graded cocommutative coalgebras and Lie algebra twisting cochains that extends the nowadays standard characterization of an ordinary sh Lie algebra (equivalently: L_∞ algebra) in terms of its associated generalized Cartan-Chevalley-Eilenberg coalgebra. Our approach avoids any higher brackets but reproduces these brackets in a conceptual manner. The new technical tool we develop is a notion of filtered multi derivation chain algebra, somewhat more general than the standard concept of a multicomplex endowed with a compatible algebra structure.

Abstract continued

The crucial observation, just as for ordinary Lie-Rinehart algebras, is this: For a general sh Lie-Rinehart algebra, the generalized Cartan-Chevalley-Eilenberg operator on the corresponding graded algebra involves two operators, one coming from the sh Lie algebra structure and the other one from the generalized action on the corresponding algebra; the sum of the two operators is defined on the algebra while the operators are individually defined only on a larger ambient algebra. We illustrate the structure with quasi Lie-Rinehart algebras. Quasi Lie-Rinehart algebras arise from foliations.

Origins and motivation

Noether theorems
Constrained systems
Batalin-Fradkin-Vilkovisky formalism
BRST
In the 1980's Stasheff started a research program aimed at developing or isolating the higher homotopies behind the formalism

Some literature

Kjesth 2001: Ph. d. thesis supervised by J. Stasheff
develops notion of sh Lie-Rinehart
published as [Kje01a], [Kje01b]
Huebschmann 2003: Quasi Lie-Rinehart algebras:
higher homotopies arising from a foliation [Hue05, Vit14, Hue17]

perhaps related with
Fredenhagen-Rejzner arxiv:1208.1428
Paugam arxiv:1106.4955

Structure of the talk

Upshot
Higher homotopy
Maurer-Cartan algebras
Lie-Rinehart algebras
Higher homotopies generalization
Multi derivation Maurer-Cartan algebras and
sh-Lie-Rinehart algebras
Quasi Lie-Rinehart algebras

For a more detailed version of this talk see

http://math.univ-lille1.fr/ huebschm/data/talks/talkbz.pdf

Upshot

A single theory having ordinary Lie algebra cohomology and ordinary de Rham cohomology as its offspring

both arise as the derived functor of the operation of taking invariants with respect to an algebra of differential operators

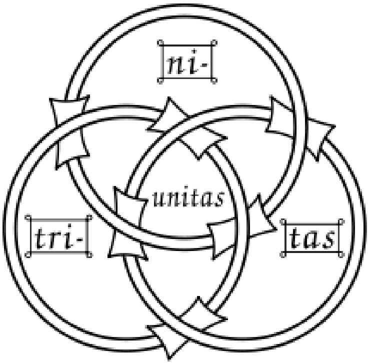
higher homotopy generalization thereof

applies e.g. to foliations: non-zero higher homotopies

Broader perspective: general gauge theory for Lie-Rinehart algebras that encompasses
classical gauge theory
differential Galois theory, in particular ordinary Galois theory
Lie theory for differential equations

Higher homotopy

Borromean rings as a symbol of the Christian Trinity, from a 13th-century manuscript



The name “Borromean rings” comes from their use in the coat of arms of the aristocratic Borromeo family in Northern Italy.

Maurer-Cartan algebras

R commutative ring with 1, A commutative R-algebra

A *Maurer-Cartan algebra* is the graded A-algebra $\mathrm{Alt}_A(L, A)$ of A-multilinear alternating forms on an A-module L, together with a differential d turning $\mathrm{Alt}_A(L, A)$ into a differential graded algebra over the ground ring R

beware: *not* in general a differential graded A-algebra
Special case $A = R$: In [VE89], van Est uses terminology *Maurer-Cartan* algebra
Example: Lie algebra $\mathfrak{g}$
$(\mathrm{Alt}(\mathfrak{g}, \mathbb{R}), d)$ ordinary Cartan-Chevalley-Eilenberg complex

Lie-Rinehart algebras

R commutative ring with 1, A commutative R-algebra
Def.: (R, A)-*Lie algebra* [Rin63]
Lie algebra L over R
$L \otimes A \to A$ left action $\vartheta \colon L \to \mathrm{Der}(A|R)$ on A by derivations
$A \otimes L \to L$ left A-module structure
compatibility conditions
generalize Lie algebra vector fields on manifold
as a module over its ring of functions

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta]$$
$$(a\alpha)(b) = a(\alpha(b))$$

for $a, b \in A$ and $\alpha, \beta \in L$

when emphasis on pair (A, L) with mutual structure of interaction
pair (A, L) : *Lie-Rinehart* algebra

Examples of Lie-Rinehart algebras

(i) M manifold, $(A, L) = (C^\infty(M), \mathrm{Vect}(M))$
(ii) A algebra, $(A, L) = (A, \mathrm{Der}(A))$
(iii) $\vartheta\colon E \to B$ Lie algebroid
(iv) Poisson algebra
(v) *twilled Lie-Rinehart algebra*
Example: M complex manifold
decomposition of complexified smooth tangent bundle
into antiholomorphic and holomorphic constituents
(vi) $\mathcal{F}$ foliation of manifold M

Lie-Rinehart algebras continued

Theorem

Given a pair that consists of a commutative algebra A and an A-module L, under suitable mild hypotheses (e. g. L finitely generated and projective as an A-module), Lie-Rinehart algebra structures on the pair (A, L) correspond bijectively to Maurer-Cartan algebra structures on $\mathrm{Alt}_A(L, A)$, that is, to operators d turning the graded A-algebra $\mathrm{Alt}_A(L, A)$ into a differential graded algebra over the ground ring R (beware: not over A)

preparation for subsequent remarks on proof:
notation: $\vartheta\colon L \to \mathrm{Der}(A|R)$ morphism of A-modules
sL suspension of L;
$t\colon sL \xrightarrow{s^{-1}} L \xrightarrow{\vartheta} \mathrm{Der}(A|R)$
"twisting cochain" when ϑ morphism of R-Lie algebras

Some remarks on the proof

R-algebra $\mathrm{Alt}(L, A)$ of A-valued R-multilin. altern. forms on L
R-linear derivations ∂^t and $\partial^{[\cdot,\cdot]}$ on $\mathrm{Alt}(L, A)$ familiar expressions

$$(\partial^t f)(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} (-1)^{(i-1)} \alpha_i (f(\alpha_1, \ldots \widehat{\alpha}_i \ldots, \alpha_n))$$

$$(\partial^{[\cdot,\cdot]} f)(\alpha_1, \ldots, \alpha_n) = \sum_{1 \le j < k \le n} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \ldots \widehat{\alpha}_j \ldots)$$

$$\mathcal{D} = \partial^t + \partial^{[\cdot,\cdot]} \colon \mathrm{Alt}(L, A) \to \mathrm{Alt}(L, A) \quad \text{derivation}$$

Proposition

When $[\cdot,\cdot]$ is Lie bracket and $\vartheta \colon L \to \mathrm{Der}(A)$ a morphism of R-Lie algebras, the derivation $\mathcal{D} = \partial^t + \partial^{[\cdot,\cdot]}$ is a differential, classical CCE operator.
When (A, L) a Lie-Rinehart algebra, derivation $\mathcal{D} = \partial^t + \partial^{[\cdot,\cdot]}$ descends to R-linear differential on $\mathrm{Alt}_A(L, A) \subseteq \mathrm{Alt}(L, A)$, even though this is not true of the individual operators ∂^t and $\partial^{[\cdot,\cdot]}$ unless $A = R$ (and ∂^t trivial).

sh Lie

$\mathfrak{g}$ graded module over the ground ring R

sh-Lie structure or *L_∞-structure on* $\mathfrak{g}$:

coalgebra differential d on $\mathrm{S}^{\mathrm{c}}[s\mathfrak{g}]$: $d = d^0 + d^1 + d^2 + \ldots$

brackets $[\,\cdot\,,\,\cdot\,]_{j+1}$

$$\begin{array}{ccc}
\mathrm{S}^{\mathrm{c}}_{j+1}[s\mathfrak{g}] & \xrightarrow{\ d^j\ } & s\mathfrak{g} \\
\uparrow & & \uparrow \\
\mathfrak{g}^{\otimes(j+1)} & \xrightarrow{[\,\cdot\,,\,\cdot\,]_{j+1}} & \mathfrak{g}
\end{array}$$

dual $\mathrm{Hom}((\mathrm{S}^{\mathrm{c}}[s\mathfrak{g}], d), R)$: generalized *Maurer-Cartan algebra*

Multi derivation chain algebra

Given: $(\mathcal{A}, \mathcal{D}_0)$ differential graded algebra
filtration

$$\mathcal{A} = \mathcal{A}^0 \supseteq \mathcal{A}^1 \supseteq \ldots \supseteq \ldots \tag{0.1}$$

compatible with differential $\mathcal{D}_0$
we say

$$(\mathcal{A}, \mathcal{D}_0, \mathcal{D}_1, \ldots) \tag{0.2}$$

multi derivation chain algebra:
family $\mathcal{D}_1, \ldots$ of derivations
for $j \geq 1$, the derivation $\mathcal{D}_j$ lowers filtration by j
$\mathcal{D} = \sum_{j \geq 1} \mathcal{D}_j$ algebra perturbation of $\mathcal{D}_0$

Multi derivation Maurer-Cartan algebras and sh-Lie-Rinehart algebras [Hue17]

A differential graded commutative algebra,
L an A-module
$\partial_{[\,\cdot\,,\,\cdot\,]} = \partial^1_{[\,\cdot\,,\,\cdot\,]} + \partial^2_{[\,\cdot\,,\,\cdot\,]} + \ldots$ degree -1 coderivation on $\mathrm{S}[sL]$
$t = t^1 + t^2 + \ldots : \mathrm{S}[sL] \longrightarrow \mathrm{Der}(A|R)$ "twisting cochain"
(A, L) *sh Lie-Rinehart algebra* compatibility conditions
$\mathrm{Sym}_A(sL, A)$: A-multilinear A-valued graded symmetric maps on sL
$\mathcal{D}_0$ algebra diff'l on $\mathrm{Sym}_A(sL, A)$ induced from diff's on A and L
induced derivations $\partial_j^{[\,\cdot\,,\,\cdot\,]}$ and ∂^{t_j} on $\mathrm{Sym}_A(sL, A)$
$\mathcal{D}_j = \partial_j^{[\,\cdot\,,\,\cdot\,]} + \partial^{t_j}$ derivation on $\mathrm{Sym}_A(sL, A)$

Theorem (Main result)

The data $(A, L, \partial_{[\,\cdot\,,\,\cdot\,]}, t)$ constitute an sh Lie-Rinehart algebra if and only if $(\mathrm{Sym}_A(sL, A), \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots)$ is a multi derivation chain algebra, necessarily the multi derivation Maurer-Cartan algebra associated with $(A, L, \partial_{[\,\cdot\,,\,\cdot\,]}, t)$.

Quasi Lie-Rinehart algebras [Hue05]

$(M, \mathcal{F})$ foliated manifold, $A = C^\infty(M)$
$\tau_{\mathcal{F}} \colon \mathrm{T}\mathcal{F} \to M$ tangent bundle to $\mathcal{F}$
$H = \Gamma(\tau_{\mathcal{F}})$: (A, H) Lie-Rinehart algebra (Frobenius)
Q a complement of H (space of sections of normal bundle)

$$\mathrm{Vect}(M) = H \oplus Q = \Gamma(\tau_{\mathcal{F}}) \oplus Q$$

$$(A, H, Q) = (C^\infty(M), L_{\mathcal{F}}, Q) \quad \textit{Lie-Rinehart triple}$$

$$(\mathcal{A}, \mathcal{Q}) = (\mathrm{Alt}_A(H, A), \mathrm{Alt}_A(H, Q)) \quad \textit{quasi-Lie-Rinehart algebra}$$

$\mathcal{A}$ "algebra of generalized functions"
$\mathcal{Q}$ "generalized Lie algebra of vector fields"
plus structure of mutual interaction, made precise shortly

$$(\mathrm{H}^*(\mathcal{A}), \mathrm{H}^*(\mathcal{Q})) \colon \quad \text{graded Lie-Rinehart algebra}$$

$$(A^H, Q^H) = (\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{Q})) \quad \text{Lie-Rinehart algebra}$$

$$\mathrm{H}^0(\mathcal{A}) \colon \quad \text{functions on space of leaves}$$

$$\mathrm{H}^0(\mathcal{Q}) \colon \quad \text{vector fields on space of leaves}$$

but $(\mathcal{A}, \mathcal{Q})$ contains more information: history

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