ENRICHED ∞-OPERADS

Hongyi Chu, joint work with Rune Haugseng and Gijs Heuts

August 2, 2017

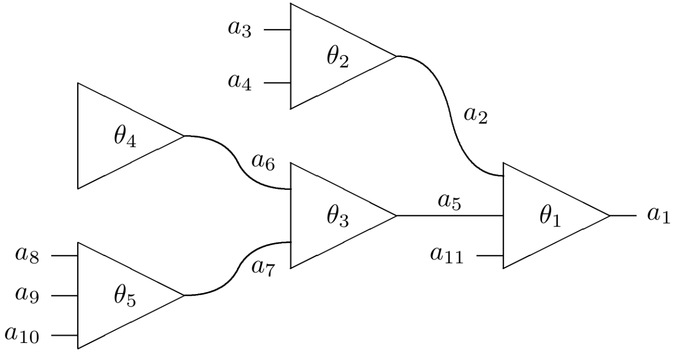
1. Operads
2. Barwick's Segal presheaves and dendroidal Segal spaces
3. Enriched ∞-operads
4. Applications

In the 70’s Boardman, Vogt and May introduced the notion of *operads* in order to study algebraic structures in topology such as loop spaces.

In the 70's Boardman, Vogt and May introduced the notion of *operads* in order to study algebraic structures in topology such as loop spaces.

Operads can be regarded as generalizations of categories where we allow morphisms to have multi-inputs.

Instead of a chain of morphisms $x_1 \to x_n \to \ldots \to x_n$, Instead of a chain of morphisms $x_1 \to x_n \to \ldots \to x_n$, we want to have



Definition (May)

A (symmetric coloured) operad $\mathcal{O}$ consists of

1. a class of objects $\mathrm{Ob}(\mathcal{O})$,

Definition (May)

A (symmetric coloured) operad $\mathcal{O}$ consists of

1. a class of objects $\mathrm{Ob}(\mathcal{O})$,
2. for finite set $\{x_i\}_{i \in I}$ and y, a set $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i \in I}; y)$,

Definition (May)

A (symmetric coloured) operad $\mathcal{O}$ consists of

1. a class of objects $\mathrm{Ob}(\mathcal{O})$,
2. for finite set $\{x_i\}_{i\in I}$ and y, a set $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}; y)$,
3. compositions

$$\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in f^{-1}(j)}; y_j) \times \mathrm{Mul}_{\mathcal{O}}(\{y_j\}_{j\in J}; z) \to \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}; z)$$

satisfy associativity and unitality conditions.

Enriched operads

Let $\mathcal{V}$ be a symmetric monoidal category. We obtain the notion of $\mathcal{V}$-*enriched operads* by requiring:

- $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_i; y) \in \mathcal{V}$,
- $\bigotimes_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in f^{-1}(j)}; y_j) \otimes \mathrm{Mul}_{\mathcal{O}}(\{y_j\}_{j\in J}; z) \to \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}; z)$ lies in $\mathcal{V}$.

1. Vect_k is a Vect_k-enriched operad with $\mathrm{Mul}_{\mathrm{Vect}_k}(V_1, ..., V_n; V) =$ vector space of n-linear maps $V_1 \times \ldots \times V_n \to V$.

1. Vect_k is a Vect_k-enriched operad with $\mathsf{Mul}_{\mathsf{Vect}_k}(V_1, ..., V_n; V) =$ vector space of n-linear maps $V_1 \times \ldots \times V_n \to V$.
2. The little n-cube operad E_n is a Top-enriched operad with one object $*$

1. Vect_k is a Vect_k-enriched operad with $\mathrm{Mul}_{\mathrm{Vect}_k}(V_1, ..., V_n; V) =$ vector space of n-linear maps $V_1 \times \ldots \times V_n \to V$.
2. The little n-cube operad E_n is a Top-enriched operad with one object $*$ and $\mathrm{Mul}_{E_n}(*_1, \ldots, *_k; *) =$ the toplogical space of configurations of k-many n-cubes within the unit n-cube in $\mathbb{R}^n$.

1. Vect_k is a Vect_k-enriched operad with $\mathrm{Mul}_{\mathrm{Vect}_k}(V_1, ..., V_n; V) =$ vector space of n-linear maps $V_1 \times \ldots \times V_n \to V$.
2. The little n-cube operad E_n is a Top-enriched operad with one object $*$ and $\mathrm{Mul}_{E_n}(*_1, \ldots, *_k; *) =$ the toplogical space of configurations of k-many n-cubes within the unit n-cube in $\mathbb{R}^n$.
3. The operad Lie is a Vect_k-eniched operad.

For operads $\mathcal{O}, \mathcal{P}$, the category of $\mathcal{O}$-*algebra in* $\mathcal{P}$ is given by $\mathrm{Fun}(\mathcal{O}, \mathcal{P})$.

For operads $\mathcal{O}, \mathcal{P}$, the category of *$\mathcal{O}$-algebra in $\mathcal{P}$* is given by $\mathrm{Fun}(\mathcal{O}, \mathcal{P})$.

Lie algebras

Lie algebras = $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Vect}_k)$.

For operads $\mathcal{O}, \mathcal{P}$, the category of $\mathcal{O}$-*algebra in* $\mathcal{P}$ is given by $\mathrm{Fun}(\mathcal{O}, \mathcal{P})$.

Lie algebras

Lie algebras = $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Vect}_k)$.

Loop spaces

May's Recognition Theorem: For X path connected:

X is equivalent to an n-fold loop space $\Omega^n Y \Leftrightarrow X \in \mathrm{Alg}_{E_n}(\mathrm{Top})$.

In the case of E_n the enrichment in spaces is important to describe compositions of paths in a topological space which are only unique up to equivalence and associative up to a coherent homotopy.

In the case of E_n the enrichment in spaces is important to describe compositions of paths in a topological space which are only unique up to equivalence and associative up to a coherent homotopy.

Such an operad is called ∞-*operad*.

In the case of E_n the enrichment in spaces is important to describe compositions of paths in a topological space which are only unique up to equivalence and associative up to a coherent homotopy.

Such an operad is called ∞-*operad*.

∞-operads are used to describe algebraic structures in categories with homotopies such as model categories, relative categories, topological categories...

But topological operads are often too rigid:

But topological operads are often too rigid:

- Weakly equivalent topological operads $\mathcal{O}$ and $\mathcal{P}$ do not need to induce equivalent homotopy theories of $\mathcal{O}$-algebras and $\mathcal{P}$-algebras.

But topological operads are often too rigid:

- Weakly equivalent topological operads $\mathcal{O}$ and $\mathcal{P}$ do not need to induce equivalent homotopy theories of $\mathcal{O}$-algebras and $\mathcal{P}$-algebras.
- Homotopy-invariant constructions such as homotopy (co)limits are difficult to set up.

But topological operads are often too rigid:

- Weakly equivalent topological operads $\mathcal{O}$ and $\mathcal{P}$ do not need to induce equivalent homotopy theories of $\mathcal{O}$-algebras and $\mathcal{P}$-algebras.
- Homotopy-invariant constructions such as homotopy (co)limits are difficult to set up.

Therefore many different approaches to homotopy-coherent operads are invented.

The category $\Delta_{\mathbb{F}}$

Let $\mathbb{F}$ be the category of finite sets, define $\Delta_{\mathbb{F}}$ as follows:

The category $\Delta_{\mathbb{F}}$

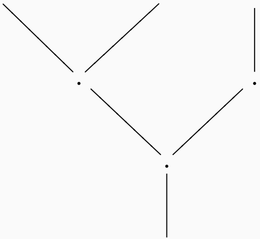
Let $\mathbb{F}$ be the category of finite sets, define $\Delta_{\mathbb{F}}$ as follows:

- $\mathrm{Ob}(\Delta_{\mathbb{F}})$: sequence of finite sets $I_0 \to I_1 \to ... \to I_m = *$.

The category $\Delta_{\mathbb{F}}$

Let $\mathbb{F}$ be the category of finite sets, define $\Delta_{\mathbb{F}}$ as follows:

- $\mathrm{Ob}(\Delta_{\mathbb{F}})$: sequence of finite sets $I_0 \to I_1 \to \ldots \to I_m = *$. Objects in $\Delta_{\mathbb{F}}$ are "levelwise" trees.



I_0={1,2,3}

I_1={1,2}

I_2=$*$

- A morphism in $\Delta_{\mathbb{F}}$

$$(I_0 \to \dots \to I_m = *) \to (J_0 \to \dots \to J_n = *)$$

is given by a map $f\colon [m] \to [n]$ in Δ together with levelwise inclusions of edges which respects the tree structure.

Definition

- We write $\mathfrak{c}_n$ for the tree $(\{1, \ldots, n\} \to *)$ and we call it a *corolla*.

Definition

- We write $\mathfrak{c}_n$ for the tree $(\{1, \dots, n\} \to *)$ and we call it a *corolla*.
- Let $\mathcal{S}$ denote the ∞-category of spaces. A *Segal presheaf* F is a presheaf $\Delta_{\mathbb{F}}^{\mathrm{op}} \to \mathcal{S}$ which satisfies the Segal condition, i.e. $F(\mathsf{k}_0 \to \dots \to \mathsf{k}_m)$ is a limit of the canonical diagram of corollas and edges in $(\mathsf{k}_0 \to \dots \to \mathsf{k}_m)$.

Definition

- We write $\mathfrak{c}_n$ for the tree $(\{1, \ldots, n\} \to *)$ and we call it a *corolla*.
- Let $\mathcal{S}$ denote the ∞-category of spaces. A *Segal presheaf* F is a presheaf $\Delta_{\mathbb{F}}^{\mathrm{op}} \to \mathcal{S}$ which satisfies the Segal condition, i.e. $F(\mathsf{k}_0 \to \ldots \to \mathsf{k}_m)$ is a limit of the canonical diagram of corollas and edges in $(\mathsf{k}_0 \to \ldots \to \mathsf{k}_m)$.
- We write $\mathrm{Seg}_{\mathbb{F}}$ for the ∞-category of Segal presheaves.

Segal presheaves encode operadic operations

1. $F(*)$ is the space of objects.

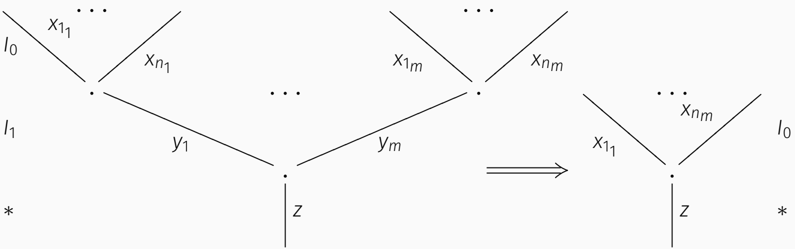
Segal presheaves encode operadic operations

1. $F(*)$ is the space of objects.
2. $F(\mathfrak{c}_n)$ is the space of n-ary multimorphisms.

Segal presheaves encode operadic operations

1. $F(*)$ is the space of objects.
2. $F(\mathfrak{c}_n)$ is the space of n-ary multimorphisms.
3. The fibre $F(\mathfrak{c}_n)_{x_1,\ldots,x_n,y}$ of the evaluation map $F(\mathfrak{c}_n) \to F(*)^{n+1}$ over $\{x_1, \ldots, x_n, y\}$ can be identified with the space of multimorphisms $\mathrm{Mul}(x_1, \ldots, x_n; y)$.

$$F(l_0 \to l_1 \to *)_{x_1,\ldots,x_{n_m},y_1,\ldots y_m,z} \qquad F(l_0 \to *)_{x_1,\ldots,x_{n_m},z}$$



By the Segal condition:

$$\left(\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I_j}; y_j)\right) \times \mathrm{Mul}_{\mathcal{O}}(\{y_j\}_{j\in J}; z) \to \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}; z).$$

- The definition of Segal presheaves is a homotopy coherent (∞-categorical) interpretation of May's definition of operads.

- The definition of Segal presheaves is a homotopy coherent (∞-categorical) interpretation of May's definition of operads.
- We need trees of "height" n to describe compositions in a Segal presheaf which are not strictly associative.

Markl's definition for operads

An operads $\mathcal{O}$ consists of a class of objects, multimorphisms and partial compositions:

Markl's definition for operads

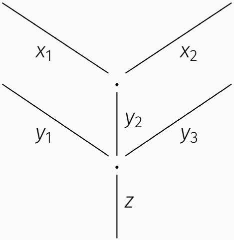
An operads $\mathcal{O}$ consists of a class of objects, multimorphisms and partial compositions:

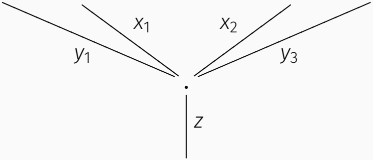
For objects $\{x_i\}_{i\in I}, \{y_j\}_{j\in J}, z$ and $j' \in J$,

$$\circ_{j'} : \ \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}, y_{j'}) \times \mathrm{Mul}_{\mathcal{O}}(\{y_j\}_{j\in J}, z) \to \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J\setminus\{j'\}}, z)$$

satisfying the unitality and associativity conditions.

(f, g) $\longrightarrow$ $f \circ_2 g$





not lw.

What is the ∞-categorical interpretation of Markl's operads?

What is the ∞-categorical interpretation of Markl's operads?

Definition (Moerdijk, Weiss)

We write Ω for the *dendroidal category* whose objects are trees and whose morphisms are generated by face maps, degeneracy maps and isomorphisms which satisfy certain dendroidal version of simplicial identities.

What is the ∞-categorical interpretation of Markl's operads?

Definition (Moerdijk, Weiss)

We write Ω for the *dendroidal category* whose objects are trees and whose morphisms are generated by face maps, degeneracy maps and isomorphisms which satisfy certain dendroidal version of simplicial identities.

We can regard Δ as a full subcategory of Ω.

Definition (Cisinski, Moerdijk)

A *dendroidal Segal space* is a functor $F\colon \Omega^{\mathrm{op}} \to \mathcal{S}$ which satisfies the Segal condition, i.e. $F(T)$ is a limit of the canonical diagram of corollas and edges in T.

Definition (Cisinski, Moerdijk)

A *dendroidal Segal space* is a functor $F\colon \Omega^{\mathrm{op}} \to \mathcal{S}$ which satisfies the Segal condition, i.e. $F(T)$ is a limit of the canonical diagram of corollas and edges in T.

We write Seg_{Ω} for the ∞-category.

Definition (Cisinski, Moerdijk)

A *dendroidal Segal space* is a functor $F\colon \Omega^{\mathrm{op}} \to \mathcal{S}$ which satisfies the Segal condition, i.e. $F(T)$ is a limit of the canonical diagram of corollas and edges in T.

We write Seg_{Ω} for the ∞-category.

The inclusions $\Delta_{\mathbb{F}} \hookleftarrow \Delta \hookrightarrow \Omega$ induces restriction functors

$$\mathrm{Seg}_{\mathbb{F}} \twoheadrightarrow \mathrm{Seg} \twoheadleftarrow \mathrm{Seg}_{\Omega},$$

where Seg denotes the ∞-category of Segal spaces.

Definition (Rezk)

A Segal space F is called *complete*, if its space of equivalences (a subspace of $F([1])$) is equivalent to $F([0])$.

Definition (Rezk)

A Segal space F is called *complete*, if its space of equivalences (a subspace of $F([1])$) is equivalent to $F([0])$.

Rezk's completion theorem

The ∞-category CSeg of complete Segal spaces is the ∞-category of small ∞-categories.

Definition (Rezk)

A Segal space F is called *complete*, if its space of equivalences (a subspace of $F([1])$) is equivalent to $F([0])$.

Rezk's completion theorem

The ∞-category CSeg of complete Segal spaces is the ∞-category of small ∞-categories.

$\Rightarrow$ We call a Segal presheaf or a dendroidal Segal space is *complete* if its underlying Segal space is complete.

Simplicial Operads

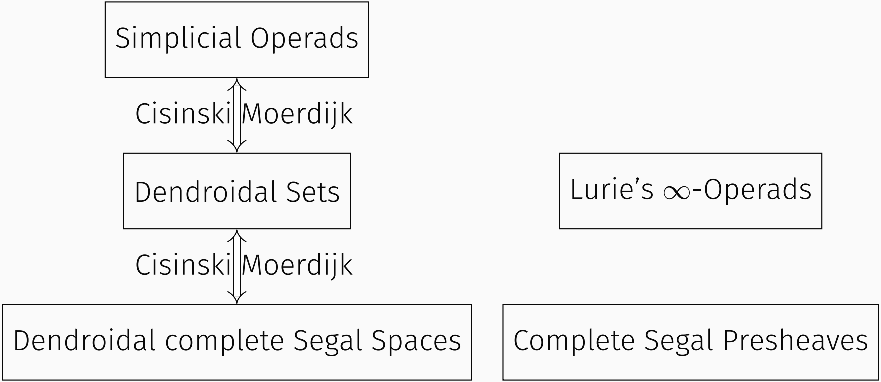
Dendroidal Sets

Lurie's ∞-Operads

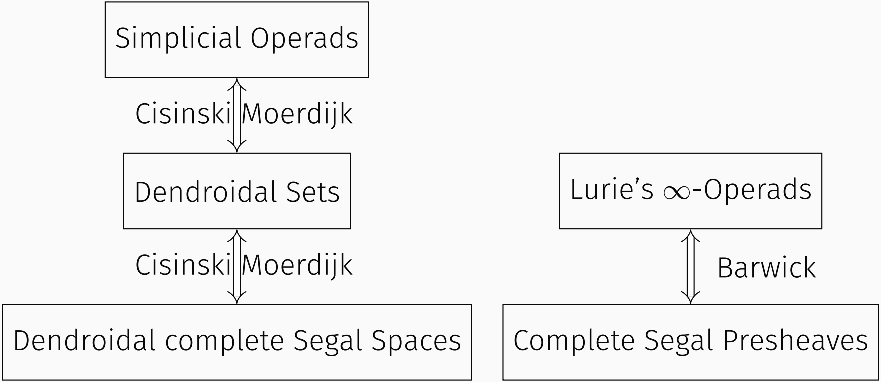
Dendroidal complete Segal Spaces

Complete Segal Presheaves

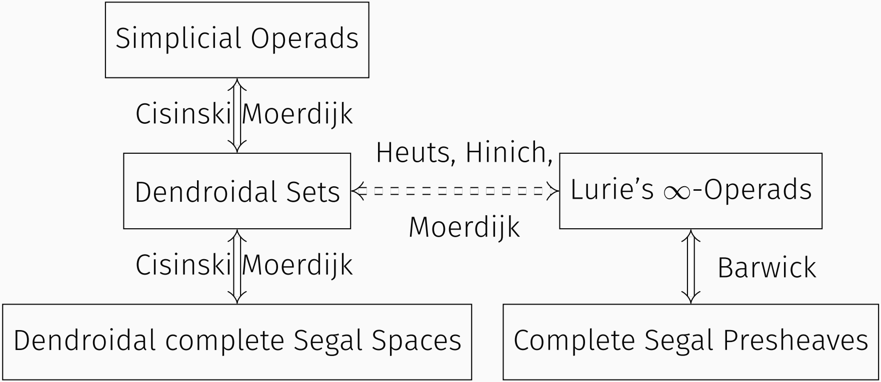
DIFFERENT MODELS FOR ∞-OPERADS

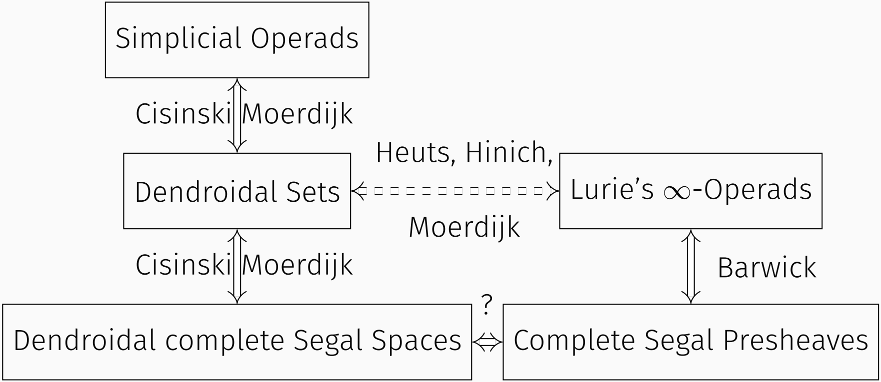


DIFFERENT MODELS FOR ∞-OPERADS



DIFFERENT MODELS FOR ∞-OPERADS




Simplicial Operads
Cisinski Moerdijk
Dendroidal Sets
Heuts, Hinich,
Moerdijk
Lurie's ∞-Operads
Cisinski Moerdijk
Barwick
?
Dendroidal complete Segal Spaces
Complete Segal Presheaves

1. For every symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, we define two different models of $\mathcal{V}$-enriched ∞-operads ($\Delta_{\mathbb{F}}$ and Ω).

HOW TO PROVE THE EQUIVALENCE?

1. For every symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, we define two different models of $\mathcal{V}$-enriched ∞-operads ($\Delta_{\mathbb{F}}$ and Ω).
2. We reprove Rezk's completion theorem in the operadic setting.

1. For every symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, we define two different models of $\mathcal{V}$-enriched ∞-operads ($\Delta_{\mathbb{F}}$ and Ω).
2. We reprove Rezk's completion theorem in the operadic setting.
3. We verify the equivalence of these two approaches.

1. For every symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, we define two different models of $\mathcal{V}$-enriched ∞-operads ($\Delta_{\mathbb{F}}$ and Ω).
2. We reprove Rezk's completion theorem in the operadic setting.
3. We verify the equivalence of these two approaches.
4. We show that we recover dendroidal Segal spaces and Segal presheaves by chosing $\mathcal{V} = \mathcal{S}$.

Definition (Segal)

A symmetric monoidal ∞-category is a functor $\mathbb{F}_* \to \mathrm{Cat}_\infty$.

Definition (Segal)

A symmetric monoidal ∞-category is a functor $\mathbb{F}_* \to \mathrm{Cat}_\infty$.

Definition

Let $\mathcal{V}_\otimes \to \mathbb{F}_*^{\mathrm{op}}$ be the fibration associated to a presentably symmetric monoidal ∞-category $\mathcal{V}$.

Definition (Segal)

A symmetric monoidal ∞-category is a functor $\mathbb{F}_* \to \mathsf{Cat}_\infty$.

Definition

Let $\mathcal{V}_\otimes \to \mathbb{F}_*^{\mathrm{op}}$ be the fibration associated to a presentably symmetric monoidal ∞-category $\mathcal{V}$. Define $\Delta_{\mathbb{F}}^{\mathcal{V}}$ by

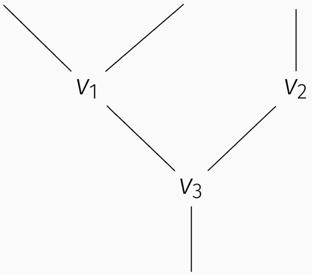
$$\begin{array}{ccc} \Delta_{\mathbb{F}}^{\mathcal{V}} & \longrightarrow & \mathcal{V}_\otimes \\ \downarrow & \lrcorner & \downarrow \\ \Delta_{\mathbb{F}} & \xrightarrow[\mathsf{Vert}]{} & \mathbb{F}_*^{\mathrm{op}}, \end{array}$$

The functor Vert carries a tree to the set of its vertices.

An object in $\Delta_{\mathbb{F}}^{\mathcal{V}}$ is of the form $((I_0 \to \ldots \to I_m = *), (v_1, \ldots, v_k))$.

An object in $\Delta_{\mathbb{F}}^{\mathcal{V}}$ is of the form $((I_0 \to \ldots \to I_m = *), (v_1, \ldots, v_k))$.

$\Rightarrow$ Objects in $\Delta_{\mathbb{F}}^{\mathcal{V}}$ are labeled trees.



Definition

A *$\mathcal{V}$-enriched Segal presheaf* is a presheaf $F\colon \Delta_{\mathbb{F}}^{\mathcal{V},\mathrm{op}} \to \mathcal{S}$ satisfying

Definition

A $\mathcal{V}$*-enriched Segal presheaf* is a presheaf $F\colon \Delta_{\mathbb{F}}^{\mathcal{V},\mathrm{op}} \to \mathcal{S}$ satisfying

1. the Segal condition as above,

Definition

A $\mathcal{V}$*-enriched Segal presheaf* is a presheaf $F\colon \Delta_{\mathbb{F}}^{\mathcal{V},\mathrm{op}} \to \mathcal{S}$ satisfying

1. the Segal condition as above,
2. the *Continous Segal condition*.

Definition

A $\mathcal{V}$*-enriched Segal presheaf* is a presheaf $F\colon \Delta_{\mathbb{F}}^{\mathcal{V},\mathrm{op}} \to \mathcal{S}$ satisfying

1. the Segal condition as above,
2. the *Continous Segal condition.* It implies that, for $v \in \mathcal{V}$, the spaces $F(\mathfrak{c}_n, v)_{x_1,\dots,x_n,x}$ determine an object $\mathrm{Mul}(x_1, \dots, x_n; x) \in \mathcal{V}$.

Definition

A $\mathcal{V}$*-enriched Segal presheaf* is a presheaf $F\colon \Delta_{\mathbb{F}}^{\mathcal{V},\mathrm{op}} \to \mathcal{S}$ satisfying

1. the Segal condition as above,
2. the *Continous Segal condition.* It implies that, for $v \in \mathcal{V}$, the spaces $F(\mathfrak{c}_n, v)_{x_1,\dots,x_n,x}$ determine an object $\mathrm{Mul}(x_1, \dots, x_n; x) \in \mathcal{V}$.

F is called complete, if its underlying Segal space is. We write $\mathrm{CSeg}_{\mathbb{F}}(\mathcal{V})$ for the corresponding ∞-category.

Definition

A map $F \to G$ of Segal presheaves is

- *fully faithful* if $F(\mathfrak{c}_n, v)_{x_1,\ldots,x_n,y} \simeq G(\mathfrak{c}_n, v)_{x_1,\ldots,x_n,y}$.

Definition

A map $F \to G$ of Segal presheaves is

- *fully faithful* if $F(\mathfrak{c}_n, v)_{x_1,\ldots,x_n,y} \simeq G(\mathfrak{c}_n, v)_{x_1,\ldots,x_n,y}$.
- *essentially surjective* if it the induced functor of the underlying ∞-categories (complete Segal spaces) is.

Definition

A map $F \to G$ of Segal presheaves is

- *fully faithful* if $F(\mathfrak{c}_n, v)_{x_1,\dots,x_n,y} \simeq G(\mathfrak{c}_n, v)_{x_1,\dots,x_n,y}$.
- *essentially surjective* if it the induced functor of the underlying ∞-categories (complete Segal spaces) is.

Completion Result

There is an equivalence of ∞-categories

$$\mathrm{CSeg}_{\mathbb{F}}(\mathcal{V}) \simeq \mathrm{Seg}_{\mathbb{F}}(\mathcal{V})[FFES^{-1}].$$

Using the same arguments:

- Define $\Omega^{\mathcal{V}}$ by pullback,

Using the same arguments:

- Define $\Omega^{\mathcal{V}}$ by pullback,
- Define the ∞-category $\mathrm{Seg}_{\Omega}(\mathcal{V})$ of $\mathcal{V}$-enriched dendroidal Segal spaces using (continuous) Segal condition.

Using the same arguments:

- Define $\Omega^{\mathcal{V}}$ by pullback,
- Define the ∞-category $\mathrm{Seg}_{\Omega}(\mathcal{V})$ of $\mathcal{V}$-enriched dendroidal Segal spaces using (continuous) Segal condition.
- Define *FFES* and prove $\mathrm{CSeg}_{\Omega}(\mathcal{V}) \simeq \mathrm{Seg}_{\Omega}(\mathcal{V})[\mathit{FFES}^{-1}]$.

Theorem

The ∞-categories $\mathrm{CSeg}_{\mathbb{F}}(\mathcal{V})$ and $\mathrm{CSeg}_{\Omega}(\mathcal{V})$ are equivalent.

Theorem

The ∞-categories $\mathrm{CSeg}_{\mathbb{F}}(\mathcal{V})$ and $\mathrm{CSeg}_{\Omega}(\mathcal{V})$ are equivalent.

(Both are the ∞-category of algebras for certain monad.)

Theorem

The ∞-categories $\mathrm{CSeg}_{\mathbb{F}}(\mathcal{V})$ and $\mathrm{CSeg}_{\Omega}(\mathcal{V})$ are equivalent.

(Both are the ∞-category of algebras for certain monad.)

Theorem

These ∞-categories are equivalent to the ∞-category of $\mathcal{V}$-enriched operads coming from classical constructions using model categories. (As long as such a model structure exists.)

Theorem

We have $\mathrm{CSeg}_{\mathbb{F}}(\mathcal{S}) \simeq \mathrm{CSeg}_{\mathbb{F}}$ and $\mathrm{CSeg}_{\Omega}(\mathcal{S}) \simeq \mathrm{CSeg}_{\Omega}$.

Theorem

We have $\mathrm{CSeg}_{\mathbb{F}}(\mathcal{S}) \simeq \mathrm{CSeg}_{\mathbb{F}}$ and $\mathrm{CSeg}_{\Omega}(\mathcal{S}) \simeq \mathrm{CSeg}_{\Omega}$.

$\Rightarrow$ All different models for ∞-operads mentioned above are equivalent.

Theorem

We have $\mathrm{CSeg}_{\mathbb{F}}(\mathcal{S}) \simeq \mathrm{CSeg}_{\mathbb{F}}$ and $\mathrm{CSeg}_{\Omega}(\mathcal{S}) \simeq \mathrm{CSeg}_{\Omega}$.

$\Rightarrow$ All different models for ∞-operads mentioned above are equivalent.

$\Rightarrow$ It provides a different proof for the fact that $\mathrm{CSeg}_{\Omega} \simeq$ simplicial operads.

- Study Koszul duality of enriched ∞-operads (joint project with Haugseng).

- Study Koszul duality of enriched ∞-operads (joint project with Haugseng).
- Define and study enriched ∞-properads or cyclic ∞-operads (joint project with Hackney).

- Study Koszul duality of enriched ∞-operads (joint project with Haugseng).
- Define and study enriched ∞-properads or cyclic ∞-operads (joint project with Hackney).

For mor details please check:

- arxiv.org/abs/1606.03826 Joint work with Haugseng and Heuts.
- arxiv.org/abs/1707.08049 Joint work with Haugseng.