Orbifolds and the Tricategory of Bimodule Categories

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1. Generalized Orbifolds

2. The Tricategory with Duals BiMod

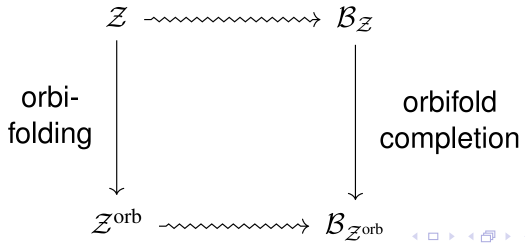
3. Finding Orbifold Data in BiMod

4. Outlook

Goal: Find Orbifold Data in the tricategory BiMod.

Generalized Orbifolds

- Generalized orbifolds were introduced by [Fröhlich, Fuchs, Runkel, Schweigert, 2009] in the context of RCFTs.
- Specifically for 2D-TQFTs with defects they were considered in [Davydov, Kong, Runkel, 2011] and then expanded on in [Carqueville, Runkel, 2012].
- Theory + symmetry $\Rightarrow$ orbifolded theory
- Symmetry not only given by groups but more generally by an orbifold datum
- This generalized notion of orbifolding also includes state sum construction internal to a given theory.
- Do this for all theories at once to obtain $\mathcal{Z} \Rightarrow \mathcal{Z}^{\text{orb}}$

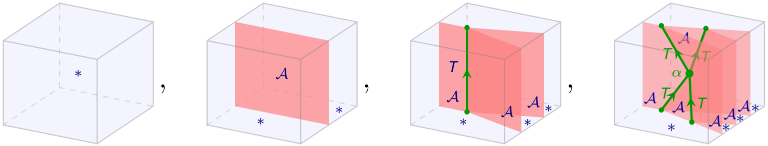


Generalized Orbifolds

- $\mathcal{B}$ pivotal bicategory. Its orbifold completion is given by
  - Objects: Pairs (a, A), where $a \in \mathcal{B}$ and A a separable symmetric Frobenius algebra on A.
  - 1-morphisms: bimodules
  - 2-morphisms: bimodule maps
- 3D TQFT $\mathcal{Z} \Rightarrow \mathcal{G}_{\mathcal{Z}}$ Gray category with duals [Carqueville, Meusburger, Schaumann 2016]
- 0-, 1-, 2-cells are labels for 3-, 2- and 1- strata. Composition is formal.
- 3-cells are obtained from $\mathcal{Z}$ by cutting out balls around points and applying $\mathcal{Z}$.

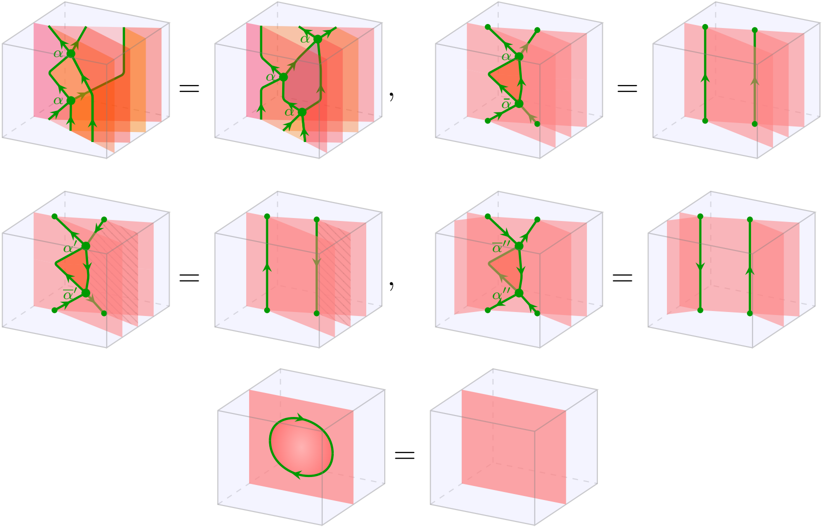
Orbifold Datum

- Orbifold Datum: Labels to decorate defect networks that are dual to triangulations such that evaluation under $\mathcal{Z}$ becomes invariant under Pachner Moves.
- One can express this internal to $\mathcal{G}_{\mathcal{Z}}$.
- Let $\mathcal{G}$ be a Gray Category with duals. A set of (special) *orbifold datum in* $\mathcal{G}$ consists of:
  - $* \in \mathcal{G}$
  - $\mathcal{A} \in \mathcal{G}(*, *)$
  - $T : \mathcal{A} \square \mathcal{A} \longrightarrow \mathcal{A}$
  - $\alpha : T \otimes (1_{\mathcal{A}} \square T) \rightleftarrows T \otimes (T \square 1_{\mathcal{A}}) : \overline{\alpha}$



such that (up to details)

Orbifold Datum



The Tricategory with Duals BiMod

- Objects: spherical fusion categories over *k*

Definition

A spherical fusion category is a *k*-linear finitely semisimple monoidal category with well behaved duality.

- 1-morphisms: *k*-linear finitely semisimple bimodule categories with bimodule traces [Schaumann, 2012]
- 2-morphisms: bimodule functors
- 3-morphisms: bimodule natural transformations
- Compositions $\square, \otimes, \circ$ of 1-, 2- and 3-morphisms
- ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}} \square {}_{\mathcal{B}}\mathcal{N}_{\mathcal{C}} = {}_{\mathcal{A}}\mathcal{M} \underset{\mathcal{B}}{\boxtimes} \mathcal{N}_{\mathcal{C}}$, the relative Deligne product
- $\otimes$ is composition of bimodule functors
- $\circ$ is composition of natural transformations

The Tricategory with Duals BiMod

- All Hom-bicategories $\mathrm{BiMod}(\mathcal{A}, \mathcal{B})$ are rigid. Duals are given by adjoint functors.
- Due to the presence of module traces they are even pivotal.
- A $\mathcal{C}$-module structure on $\mathcal{M}$ ($\mathcal{C}$ a fusion category) induces a $\mathcal{C}$-enrichment on $\mathcal{M}$ [Ostrik, 2001]. The inner Hom $\langle -, - \rangle$ is determined by an adjunction

$$\mathcal{M}(c \triangleright x, y) \cong \mathcal{M}(c, \langle x, y \rangle).$$

- The dual of a bimodule category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is given by ${}_{\mathcal{B}}\mathcal{M}^{op}{}_{\mathcal{A}}$, with the inner Hom "pairing"

$$\langle -, - \rangle : {}_{\mathcal{A}}\mathcal{M} \underset{\mathcal{B}}{\boxtimes} \mathcal{M}^{op}{}_{\mathcal{A}} \longrightarrow \mathcal{A}.$$

The "copairing" is given by the adjoint functor.

Theorem (Carqueville, Runkel, Schaumann, unpublished)

A spherical fusion category $\mathcal{C}$ gives rise to an orbifold datum in BiMod in the following way.

- $* =$ Vect
- $\mathcal{A} = \mathcal{C}$
- $T = \otimes : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathcal{C}$
- $\alpha =$ *associator,* $\overline{\alpha} = \alpha^{-1}$ *(up to a scalar)*

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  - Conversely, this allows one to think of an orbifold datum in a Gray category $\mathcal{G}$ with duals as a (generalized) spherical fusion category internal to $\mathcal{G}$.

Theorem (S., unpublished)

An inclusion of fusion categories $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ gives rise to an orbifold datum in BiMod.

- $* = \mathcal{C}_0$
- $\mathcal{A} = \mathcal{C}$ *(as $\mathcal{C}_0$-$\mathcal{C}_0$ bimodule)*
- T = *induced from* $\otimes$
- α = *induced from associator,* $\overline{\alpha} = \alpha^{-1}$ *(up to a scalar)*

3D Group Orbifolds

- G a finite group
- $\rho : \mathbf{B}G \longrightarrow$ BiMod a 3-functor with $\rho(*) = \mathcal{C}_0$
- $\mathcal{A}_\rho = \bigoplus_\rho \rho(g)$
- Structure morphisms of ρ endow $\rho(1)$ and $\mathcal{A}_\rho$ with the structures of monoidal categories such that $\rho(1) \cong \mathcal{C}_0$ monoidally.
- If A_ρ is a fusion category then it is a G-extension of $\rho(1)$ in the sense of [Etingof, Gelaki, Nikshych, Ostrik].
- If A_ρ is in addition spherical then we have an inclusion $\mathcal{C}_0 = \rho(1) \hookrightarrow \mathcal{A}_\rho = \mathcal{C}$ and the theorem applies.

The Relative Deligne Product

Definition

Let $\mathcal{M}, \mathcal{N}$ be semisimple categories. A Deligne product of $\mathcal{M}$ and $\mathcal{N}$ consists of a semisimple category $\mathcal{M} \boxtimes \mathcal{N}$ and a bilinear functor $D : \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{M} \boxtimes \mathcal{N}$ such that

$$D^* : \mathrm{Fun}^{\mathrm{lin}}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{K}) \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{bilin}}(\mathcal{M} \times \mathcal{N}, \mathcal{K})$$

for all $\mathcal{K}$ (together with adjoint equivalence data for D^*).

- The Deligne Product of semisimple categories always exists and has the following description.
  - Objects: formal sums of pairs $m \boxtimes n$ where $m \in \mathcal{M}, n \in \mathcal{N}$
  - $\mathcal{M} \boxtimes \mathcal{N}(m \boxtimes n, m' \boxtimes n') = \mathcal{M}(m, m') \otimes \mathcal{N}(n, n')$.
  - Simple objects are given by pairs $x \boxtimes y$ for x, y simple.

The Relative Deligne Product

Definition

- $F : \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N} \longrightarrow \mathcal{K}$ is balanced if it comes equipped with a natural isomorphism

$$F(m \triangleleft c \boxtimes n) \xrightarrow{\cong} F(m \boxtimes c \triangleright n)$$

(that satisfies some coherence).

- For $\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{N}$ module categories over $\mathcal{C}$, a relative Deligne Product $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ consists of a functor $B : \mathcal{M} \boxtimes \mathcal{N} \longrightarrow \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ such that

$$B^* : \mathrm{Fun}(\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}, \mathcal{K}) \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{bal}}(\mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}, \mathcal{K})$$

for any $\mathcal{K}$ (plus a choice of adjoint equivalence data for B^*).

The Relative Deligne Product

Definition

- $G : \mathcal{L} \longrightarrow \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}$ is cobalanced if it comes equipped with a natural isomorphism

$$G(x) \triangleleft c \xrightarrow{\cong} c \triangleright G(x)$$

(satisfying coherence).

- For $\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{C}}\mathcal{N}$ module categories over $\mathcal{C}$, a corelative Deligne Product $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ consists of a functor $U : \mathcal{M} \boxtimes \mathcal{N} \longrightarrow \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ such that

$$U_* : \mathrm{Fun}(\mathcal{L}, \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}) \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{cobal}}(\mathcal{L}, \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N})$$

for any $\mathcal{L}$ (plus a choice of adjoint equivalence data for U_*).

The Relative Deligne Product

- F is balanced iff its adjoint is cobalanced.
- $B : \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N} \longrightarrow \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ is a relative Deligne product iff its adjoint $U : \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N} \longrightarrow \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}$ is a corelative Deligne product.

The Relative Deligne Product

- F is balanced iff its adjoint is cobalanced.
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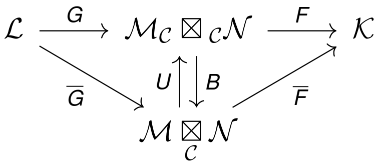
$\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ can be realized in several ways.

- View $\mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}$ as left $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$-module.
- $A = \bigoplus_{u}^{*} u \boxtimes u \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ carries a canonical structure of a separable symmetric Frobenius algebra.
- A (co)relative Deligne product is given by

$$B : \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N} \rightleftarrows A\text{-Mod}(\mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}) : U$$

where B is the free and U the forgetful functor.

Relative vs Non-Relative Product



- The adjunction $U \vdash B$ induces a separable symmetric Frobenius monad $\Lambda = U \otimes B$ on $\mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}$ given by $\Lambda : X \longmapsto A \triangleright X$.
- There are equivalences of categories

$$\mathrm{Fun}^{\mathrm{bal}}(\mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}, \mathcal{K}) \cong \text{Mod-}\Lambda(\mathrm{Fun}(\mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}, \mathcal{K}))$$

  and

$$\mathrm{Fun}^{\mathrm{cobal}}(\mathcal{L}, \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}) \cong \Lambda\text{-Mod}(\mathrm{Fun}(\mathcal{L}, \mathcal{M}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\mathcal{N}))$$

Relative vs Non-Relative Product

- $\overline{F} \otimes \overline{G}$ can be written as a coequalizer

$$F \otimes \Lambda \otimes G \rightrightarrows F \otimes G \xrightarrow{\pi} \overline{F} \otimes \overline{G}$$

  and as an equalizer

$$F \otimes \Lambda \otimes G \leftleftarrows F \otimes G \xleftarrow{u} \overline{F} \otimes \overline{G}$$

  such that $\pi u = 1$.
- Using this and similar considerations one can relate the various compositions of cells defined on relative Deligne products to their counterparts with extra structure defined on ordinary Deligne products.
- This allows to reduce the orbifold equations to ones in $\mathcal{C}$.

Outlook

- Eventually one would want to have an orbifolding completion for Gray categories with duals: $\mathcal{G} \longmapsto \mathcal{G}_{\text{orb}}$
- The above result should then be obtained as a special case of $(\mathcal{G}_{\text{orb}})_{\text{orb}} \cong \mathcal{G}_{\text{orb}}$.