Orbifolds of defect TQFTs

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Goal.

Generalise and unify **orbifold** and **state sum constructions**

Slogans.

- "State sum models = orbifolds of the trivial theory"
- "General orbifolds = state sum constructions internal to some QFT"

Result.

Worked out for any n-dimensional **defect TQFT**

Applications.

- new equivalences between singularity categories
- "spherical fusion categories internal to Gray categories with duals"
- surface defects in Reshetikhin-Turaev theory
- topological quantum computation(?)

2-dimensional closed TQFT

A **2-dimensional closed TQFT** is a symmetric monoidal functor

$$\mathcal{Z} \colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}_{\mathbb{k}}$$

Atiyah 1988

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Examples.

- **Dijkgraaf-Witten models**:
  $\mathbb{k}[G]$ for finite abelian group G
- **Sigma models**:
  $H_d(M)$ for compact oriented manifold M
- **Landau-Ginzburg models**:
  $\mathbb{C}[x_1, \dots, x_n]/(\partial_x W)$ for isolated singularity $W \in \mathbb{C}[x_1, \dots, x_n]$

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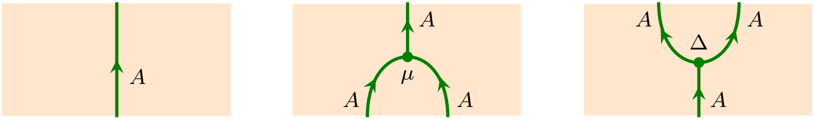
- **State sum models**

Atiyah 1988

State sum models in 2d

- input: separable symmetric Frobenius $\mathbb{k}$-algebra (A, μ, Δ)
- choose oriented **triangulation** for every bordism Σ
- **decorate Poincaré dual** graph with (A, μ, Δ):



Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

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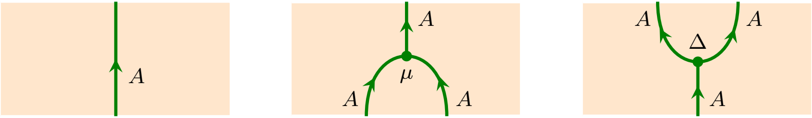

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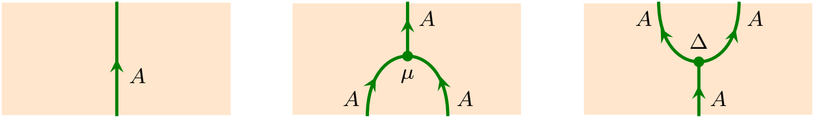

- associate $A^{\otimes k}$ to dual triangulation of circle with k points
- get maps $\pi_\Sigma \colon A^{\otimes k_1} \otimes \cdots \otimes A^{\otimes k_m} \longrightarrow A^{\otimes l_1} \otimes \cdots \otimes A^{\otimes l_n}$ from bordism $\Sigma \colon (S^1)^{\times m} \longrightarrow (S^1)^{\times n}$

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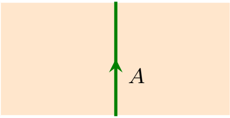

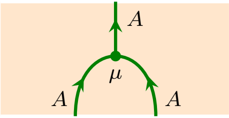

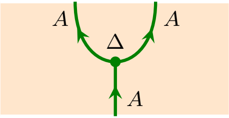

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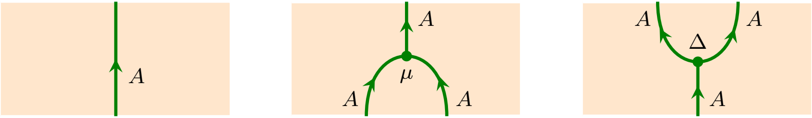

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- if $\Sigma \colon S^1 \longrightarrow S^1$ is cylinder, then $\pi_\Sigma \colon A^{\otimes k} \longrightarrow A^{\otimes k}$ is projector
- define **state sum model**

$$\mathcal{Z}_A^{\mathrm{ss}} \colon \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}_\Bbbk$$

$$S^1 \longmapsto \mathrm{Im}\Big(\pi_{S^1 \times [0,1]} \colon A^{\otimes k} \longrightarrow A^{\otimes k}\Big) \cong Z(A)$$

$$\Big(\Sigma \colon (S^1)^{\times m} \longrightarrow (S^1)^{\times n}\Big) \longmapsto \Big(\text{induced linear map } Z(A)^{\otimes m} \longrightarrow Z(A)^{\otimes n}\Big)$$

Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

State sum models in 2d

Theorem.
State sum model for A is independent of choice of triangulation, and $\mathcal{Z}^{\mathrm{ss}}_A(S^1) \cong Z(A)$.

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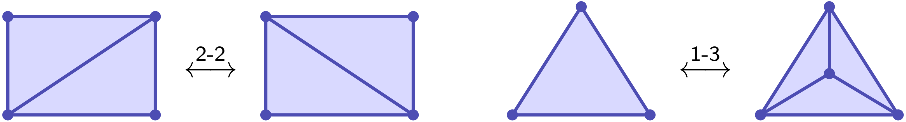

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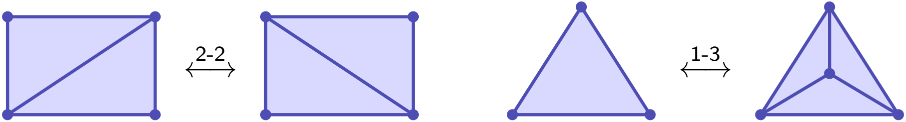

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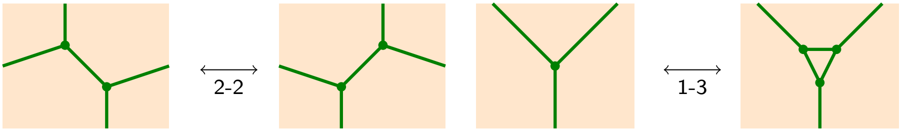

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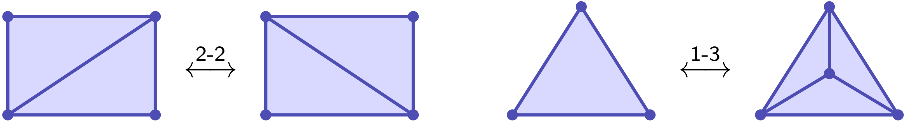

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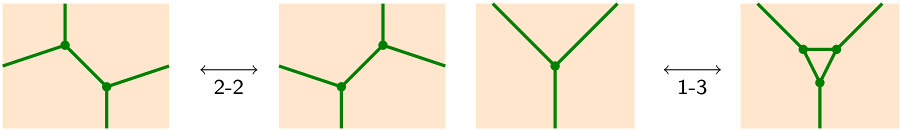

Satisfied for separable symmetric Frobenius $\mathbb{k}$-algebras A!

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2-dimensional defect TQFT

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$$\mathcal{Z}\colon\ \mathrm{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathrm{Vect}_{\Bbbk}$$

Davydov/Kong/Runkel 2011

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- a set D_2 to label 2-strata of bordisms
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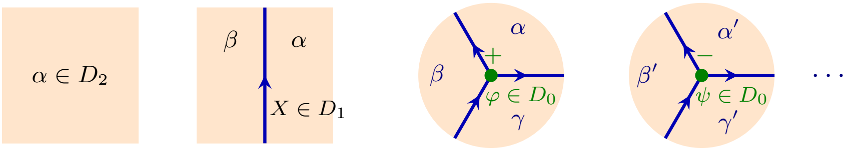

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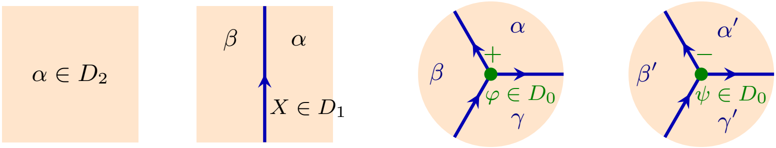
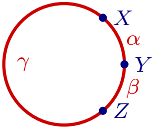
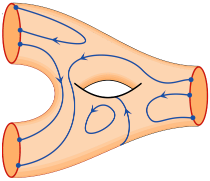


$\cdots$

objects: 

morphisms: 

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Examples of 2d defect TQFTs

- **A-models**: symplectic manifolds & Fukaya categories (conj.)
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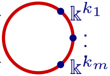
  - $\mathcal{Z}^{\text{triv}}\Bigg($ 

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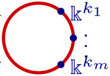
  - $\mathcal{Z}^{\mathrm{triv}}\Big($ 

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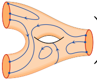
  - $\mathcal{Z}^{\mathrm{triv}}\Big($  $\Big) \stackrel{\mathrm{def}}{=}$ (evaluate line and point defect in $\mathrm{Vect}_{\Bbbk}$)

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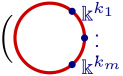
  - $\mathcal{Z}^{\text{triv}}\Bigg($ 

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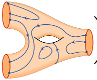
  - $\mathcal{Z}^{\text{triv}}\Bigg($  $\Bigg) \overset{\text{def}}{=}$ (evaluate line and point defect in $\operatorname{Vect}_{\Bbbk}$)
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Orbifolds

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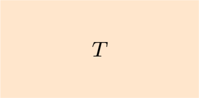
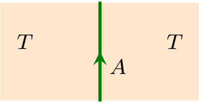
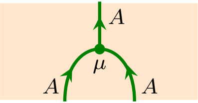
An **orbifold datum** for $\mathcal{Z}$ is $\mathcal{A} \equiv (T, A, \mu, \Delta)$:



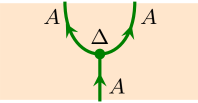
$T \in D_2$



$A \in D_1$



$\mu \in D_0$



$\Delta \in D_0$

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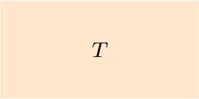
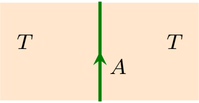
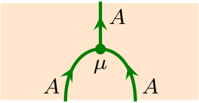


$T \in D_2$



$A \in D_1$



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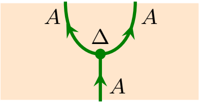

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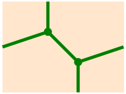
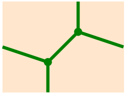
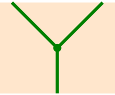
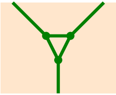
such that Pachner moves are identities under $\mathcal{Z}$:

$\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

$\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

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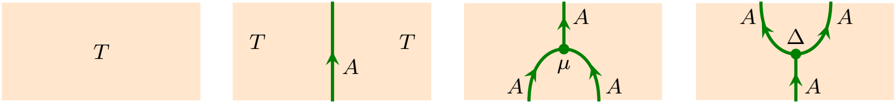

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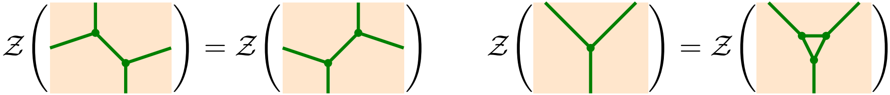

Definition & Theorem.
Applying $\mathcal{Z}$ to $\mathcal{A}$-decorated dual triangulations gives **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}}\colon \operatorname{Bord}_2 \longrightarrow \operatorname{Vect}_{\Bbbk}$$

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Orbifold equivalence

Theorem.
2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{B}_{\mathcal{Z}}$

Davydov/Kong/Runkel 2011

Orbifold equivalence

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Definition & Lemma.
The **orbifold completion** of a pivotal 2-category $\mathcal{B}$ is the pivotal 2-category $\mathcal{B}_{\mathrm{orb}} \cong (\mathcal{B}_{\mathrm{orb}})_{\mathrm{orb}}$ with:

- *objects* are **separable symmetric Frobenius algebras** $A \in \mathcal{B}(\alpha, \alpha)$.
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Corollary.
$\alpha, \beta \in \mathcal{B}$ orbifold equivalent via $X \implies \mathcal{B}(\gamma, \beta) \cong \mathrm{mod}_{X^\dagger \otimes X}\mathcal{B}(\gamma, \alpha)$
Davydov/Kong/Runkel 2011, Carqueville/Runkel 2012

Examples of 2d orbifolds

- **group orbifolds**: $\mathcal{Z}^G = \mathcal{Z}_{A_G}$, with $A_G = \bigoplus_{g \in G} \rho(g)$ from group action $\rho\colon G \longrightarrow \mathcal{B}_{\mathcal{Z}}(\alpha, \alpha)$

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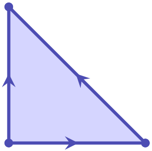
In any dimension $n \geqslant 1$, the **generalised orbifold construction** *works for any n-dimensional defect TQFT*

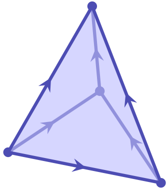
$$\mathcal{Z} \colon \operatorname{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \operatorname{Vect}_{\Bbbk} .$$

Carqueville/Runkel/Schaumann 2017

Triangulations

- **standard n-simplex** $\Delta^n := \left\{ \sum_{i=1}^{n+1} t_i e_i \;\middle|\; t_i \geqslant 0\,,\ \sum_{i=1}^{n+1} t_i = 1 \right\} \subset \mathbb{R}^{n+1}$





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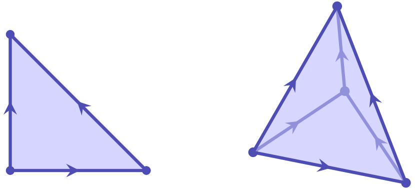

- **simplicial complex** C is finite collection of simplices such that
  - all faces of all $\sigma \in C$ are also in C
  - $\sigma, \sigma' \in C \quad \Longrightarrow \quad \sigma \cap \sigma' = \emptyset$ or $\sigma \cap \sigma' =$ face
- **triangulation** of manifold M is simplicial complex C with homeomorphism $|C| \longrightarrow M$
- (details for smooth, oriented, . . .)

Pachner moves

Let $F \subset \partial\Delta^{n+1}$ be collection of n-simplices. Let M be triangulated manifold with $K \subset M$ such that $K \overset{\varphi}{\cong} F$.

A **Pachner move** "glues the other side of $\partial\Delta^{n+1}$ into M":

$$M \longmapsto \big(M \setminus K\big) \cup_{\varphi|_{\partial K}} \big(\partial\Delta^{n+1} \setminus \mathring{F}\big)$$

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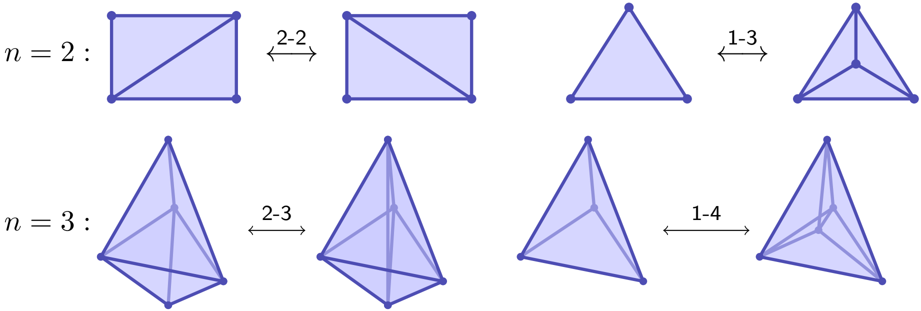

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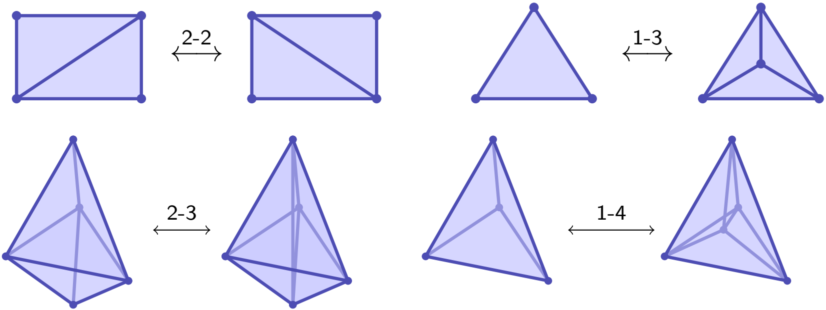

Theorem.
If triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves between them.
Pachner 1991

n-dimensional defect TQFT

An ***n*-dimensional defect TQFT** is a symmetric monoidal functor

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Carqueville/Runkel/Schaumann 2017

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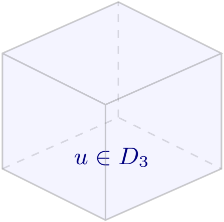

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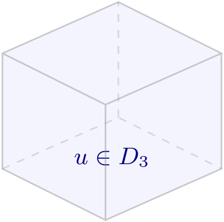

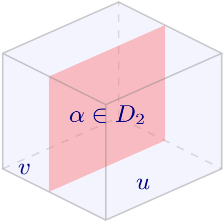

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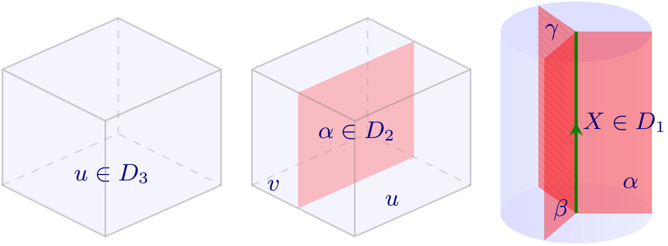

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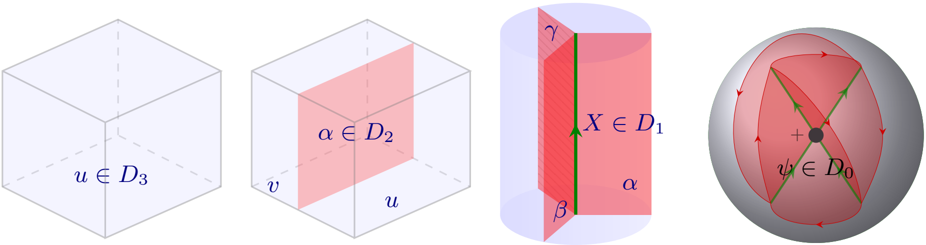

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Carqueville/Runkel/Schaumann 2017

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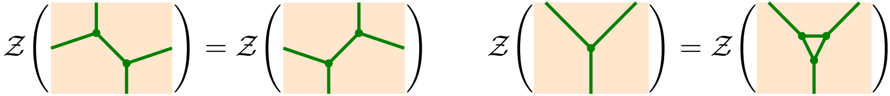
Recovers case $n = 2$:



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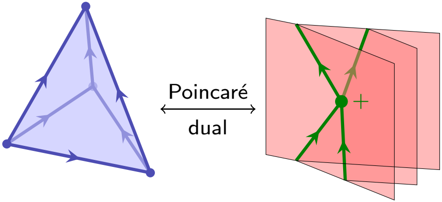
Orbifold datum $\mathcal{A}$ for $n = 3$



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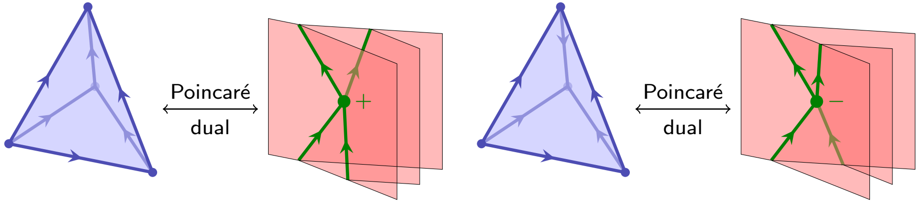

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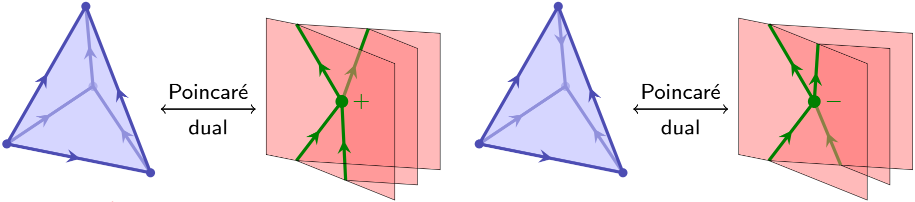

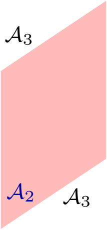

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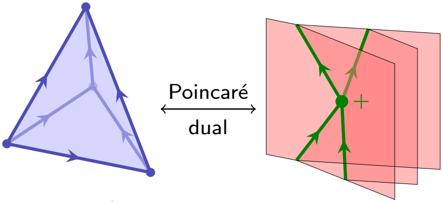

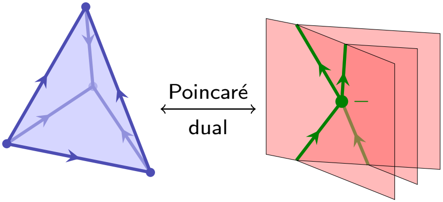

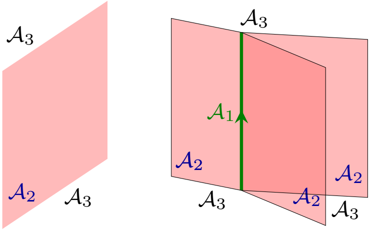

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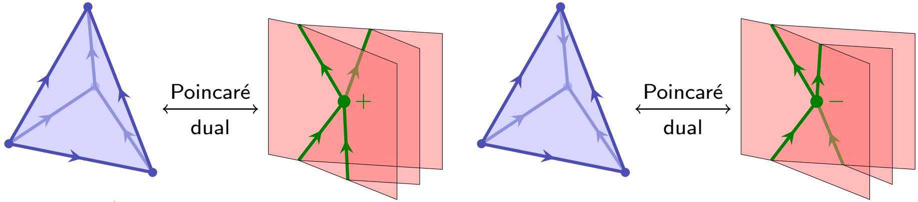

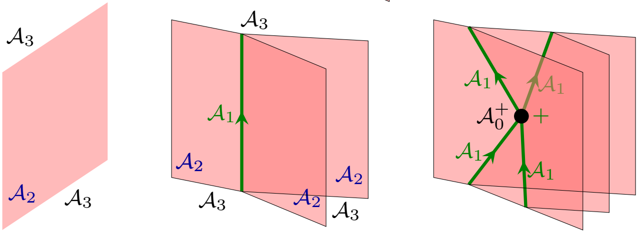

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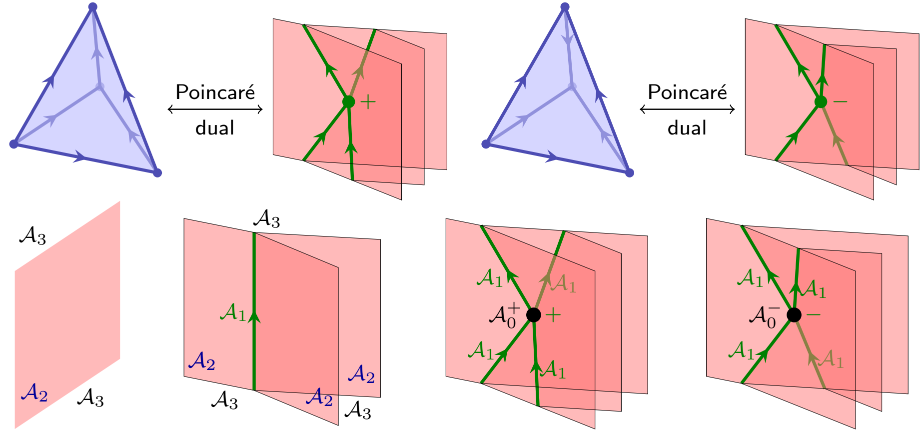

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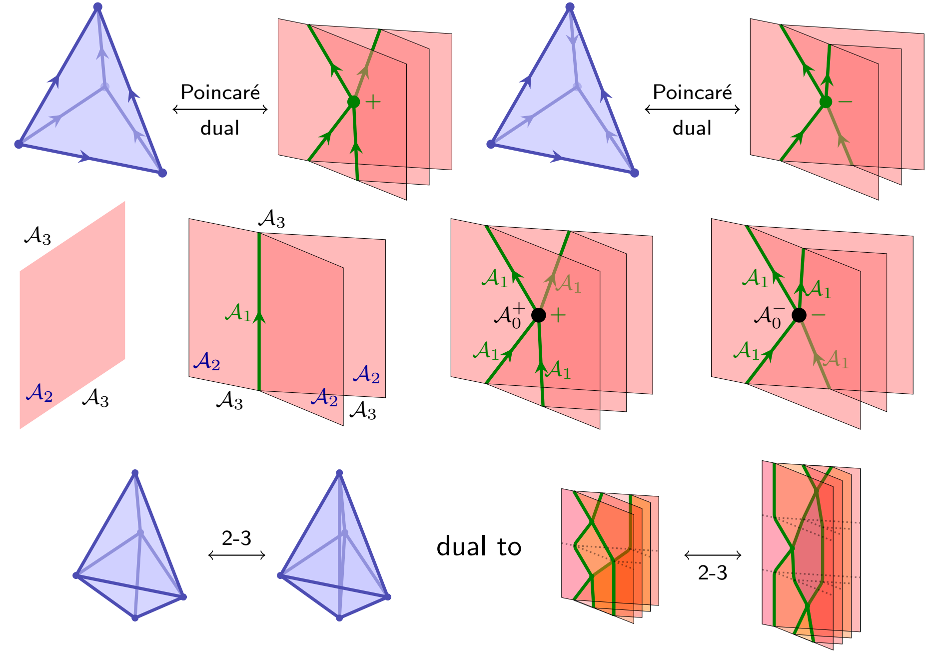

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Carqueville/Runkel/Schaumann 2017

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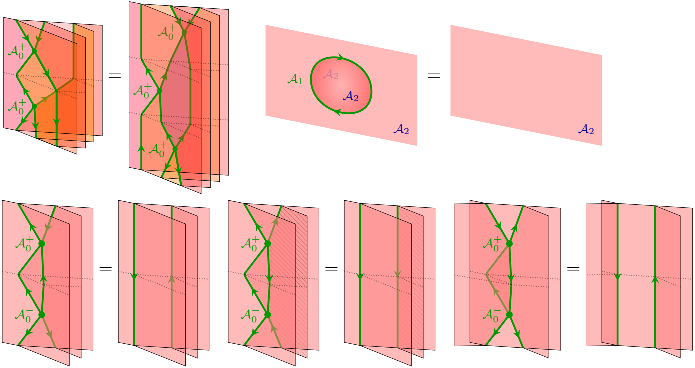

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Carqueville/Runkel/Schaumann 2017–201?, Scherl 2017, Passegger 201?