HQFTs and Beyond

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Overview

1. Recall of simplicial groups and $\mathcal{S}$-groupoids
2. TQFTs
3. Turaev's HQFTs
4. Relative TQFTs v. HQFTs
5. Homotopy finite G
6. Singular manifolds ... towards defects

$\mathcal{S}-Grpds$ = simplicially enriched groupoids.

- $\mathcal{G} : \mathcal{S} \to \mathcal{S}-Grpds$, Dwyer-Kan loop groupoid functor.
- The functor $\mathcal{G}$ has a left adjoint, $\overline{W}$.
- For any $\mathcal{S}$-groupoid, $\mathbb{G}$, $\overline{W}\mathbb{G}$ is a Kan complex (and if $\mathbb{G}$ is finite one can count the fillers for any given horn). Question: can the role of $\overline{W}\mathbb{G}$ in later slides be generalised to being a quasi-category having finitely many fillers for each inner horn? (This may be useful for handling the case of defect TQFTs.)
- These functors give an equivalence of homotopy categories and $\overline{W}\mathbb{G}$ is a 'classifying space' for principal $\mathbb{G}$-bundles.

Moore complex

For a simplicial group, or $\mathcal{S}$-groupoid, G, its Moore complex is defined to be the chain complex:

$$NG_n = \bigcap_{i=1}^{n} Ker\ d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_0^n by restriction.

Truncated simplicial groups and and links with n-groups.

We often consider Moore complexes that are *truncated* in the sense that there is some $n \geq 1$ such that $NG_k = 1$ for all $k > n$.

If $NG_k = 1$ for all $k \geq 1$, then G is a constant simplicial group (so is really just a group).
If $NG_k = 1$ for all $k \geq 2$, then $NG_1 \stackrel{\partial}{\to} NG_0$ is a crossed module, so 'is' a 2-group.

If $NG_k = 1$ for all $k \geq 3$, then $NG_2 \stackrel{\partial}{\to} NG_1 \stackrel{\partial}{\to} NG_0$ is a 2-crossed module / 3-group. It has a pairing

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

which 'lifts' the interchange law (which is thus not assumed to hold) making the difference of the two sides into a boundary.

Thin elements

Keypoint: The product of degenerate elements need not be degenerate:
e.g. $x, y \in NG_1$ then $[s_0x, s_1y][s_1y, s_1x]$ need not be degenerate. It is the lift, $x, y \in NG_2$, so is the obstruction to interchange in the corresponding n-group.
Such elements will be called 'thin' elements.

Form D_n the subgroup of G_n generated by these.

In general, G corresponds to a strict infinity groupoid if $NG_n \cap D_n = \{1\}$ for all $n \geq 1$, i.e., in general, the elements of D_n give where the 'weakness' of the infinity groupoid resides!
Strict infinity groupoid $=$ horns have unique thin fillers.

... and the thin filtration of $\overline{W}G$

Truncate G at level n, and then generate up to get the n-skeleton, sk_nG, of G. We have $(sk_nG)_m \subseteq D_m$ for $m > n$ and the skeletal filtration of G.

This also gives a filtration, $\underline{F}(G) := \{F_n(\overline{W}G) \mid n \geq 0\}$, of $\overline{W}G$, that we call the thin filtration, so

$$F_n(\overline{W}G) = \overline{W}sk_{n-1}G.$$

(Each of the $F_n(\overline{W}G)$ is a Kan complex, and in fact explicit algorithmic fillers can be given; see the Menagerie notes, [7].)

TQFTs

PL or smooth orientable $(d-1)$-manifolds and cobordisms between them form a category, $d-Cob$, with some technical reservations

Definition: A TQFT is a monoidal functor, $Z : d-Cob \to Vect^{\otimes}$, so Z preserves $\otimes$ and $Z(\emptyset) = \mathbb{C}$.

We could replace $Vect^{\otimes}$ by any suitably structured symmetric monoidal category, or more generally ... .

Building TQFTs: the Yetter models

A very quick cut-down overview:

(Yetter 1992): Fix a finite group, G, and let X be a space with triangulation, **T**.
Order the vertices of T so as to get a simplicial set.

Definition: (Yetter, [12], 1992) A *G-colouring* of **T** is a map,

$$\lambda : T_1 \to G,$$

such that given $\sigma \in T_2$, $\lambda(e_1)^{\varepsilon_1}\lambda(e_2)^{\varepsilon_2}\lambda(e_3)^{\varepsilon_3} = 1$, where the boundary, $\partial\sigma$, of σ is given by $\partial\sigma = e_1^{\varepsilon_1} e_2^{\varepsilon_2} e_3^{\varepsilon_3}$.

See also Yetter, [13], in which he used crossed modules in place of finite groups.

Draw a picture of a 2-simplex suitably ‘coloured’:

We write $\Lambda_G(\mathbf{T})$ for the set of such G-colourings and $Z_G(X, \mathbf{T})$ for the vector space with basis labelled by $\Lambda_G(\mathbf{T})$.

1) *Important:* a G-colouring of **T** is equivalent to a morphism

$$\lambda : \mathcal{G}(T) \to K(G, 0)$$

from the Dwyer-Kan loop groupoid on T to the constant finite simplicial group on G.

Equivalently λ goes from T to $\overline{W}(K(G, 0))$, which leads to a bundle theoretic interpretation of G-colourings.

This suggests to replace 'G a finite group' by 'G a finite simplicial group' and thus $K(G, 0)$ just by G (and it works, TP, [5, 6], 1998). We will assume this from now on.

Question for discussion: We know a lot about simplicial groups, G, but how does that knowledge help with studying $\overline{W}(G)$ and the structure of the *simplicial set* of 'colourings' from T to $\overline{W}G$?

2) If $\mathbf{T}'$ is a subdivision of $\mathbf{T}$, composition with a map, $r_{\mathbf{T}}^{\mathbf{T}'}$, coming from some strong deformation retraction data relating $\mathcal{G}(\mathbf{T})$ and $\mathcal{G}(\mathbf{T}')$, induces a function,

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T}),$$

which extends to a linear map from $Z_G(X, \mathbf{T}')$ to $Z_G(X, \mathbf{T})$.

Let $Z_G(X) = colim_{\mathbf{T}} Z_G(X, \mathbf{T})$. This vector space is finite dimensional and defines the 'object mapping' part of the functor Z_G.
Known in detail only for low dimensions as yet: $Z_G(X)$ has a basis in bijection with $[\underline{T}, \underline{F}(G)]^{filt}$, the set of *filtered* homotopy classes of filtered maps from the skeletal filtration of T to the thin filtration of $\overline{W}(G)$ for any triangulation T.

3) If $(M, \mathcal{T})$ is a triangulated cobordism from $(X, \mathbf{T})$ to $(Y, \mathbf{S})$, then define a linear map, $Z^!_G(M, \mathcal{T})$, by: for $\lambda \in \Lambda_G(\mathbf{T})$,

$$Z^!_G(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_G(\mathcal{T}) \\ \mu|\mathbf{T} = \lambda}} \mu|\mathbf{S}.$$

These maps will not respect composition so need normalising / averaging over possible choices. Details omitted, see [5, 6].

Could we have a simplicial vector space structure here and if so what would the averaging process correspond to?

Some thoughts:

For a (2+1) TQFT, the manifolds are surfaces, and the cobordisms 3-manifolds.
A G-colouring of a triangulation, T, of a 2-manifold, X, is a morphism, $\lambda : T \to \overline{W}G$. As T is coming from a triangulation of a 2-manifold, it equals its own 2-skeleton, so λ does not involve more than the bottom few layers of NG. Colourings of cobordisms will involve one more layer of NG.

Is the 'weak' structure (interchange lifting, etc.) observable in (some variant of) the corresponding TQFT?

HQFTs

HQFTs

Problem : would like to have a theory with manifolds **with extra structure**, e.g. a given G-bundle, metric etc.

Suggestion by Turaev, [9, 10] (1999): Replace 'just a manifold', X, by 'X, together with a characteristic structure map, $g : X \to B$', where B is some 'background' space, for instance, $B = BG$, the classifying space of a group, G.
see also Turaev's book: [11].

Similar idea explored by Lurie, [1], (2009), for extended TQFTs.

For the cobordisms, want $F : M \to B$ agreeing with the structure maps on the ends, but F will only be given 'up to homotopy relative to the boundary', (suggests a truncation of something ∞-groupoidal).

Get a monoidal category $d-Hocobord(B)$: (Rodrigues, [8], 2000) and Turaev's HQFTs translate to:

Proposition: A HQFT is a monoidal functor,

$$\tau : d-Hocobord(B) \rightarrow Vect.$$

Generation of simplicial HQFTs (work in progress, some details still to explore).

This is a sketch of a 'madcap idea for continuing investigation'.
Let $\varphi : G \to H$ be an epimorphism of simplicial groups having a finite kernel.
Several geometric structures can be encoded in somewhat this way, up to homotopy, e.g. Spin structures, comparison of PL and Top structures via microbundles[1].

[1] An old source is Milnor, [4], and more recently there are Lurie's course notes, [3].

One can adapt the notion of Yetter's colourings to take values in BG, but relative to a fixed H-colouring, and to work with manifolds over BH as if for a HQFT. This does give a sort of 'relative TQFT', (but may not fully give a HQFT, still to be examined). The interpretation would be given a fixed piece of 'extra H-structure' on X with a classification of the possible change of group to G-structures.

For both the Yetter model with finite simplicial group G, and the corresponding HQFTs, using instead a homotopy finite simplicial group (i.e., representing a homotopy n-type for some n and having finite homotopy groups) would be an interesting and useful extension (but seems quite hard to do).

More generally, having used a G as coefficients for a Yetter model TQFT, can one induce nice transformations from 'change of coefficients' along a morphism of simplicial groups?

Yetter models for defect TQFTs and HQFTs?

(i) Examples of defect TQFTs have been given e.g. by Carqueville, Runkel, and Schauman and connections with HQFTs explored in work by Carqueville, Meusburger and Schaumann (2016).

(ii) For A a poset, Ayala, Francis, and Tanaka define an A-stratified space (following Lurie in *Higher algebra*) as a space, X, together with a continuous map to A, considered as topological space. An important condition is a 'conical stratification' condition. Examples include simplicial complexes, and ...

Orbifolds:

(iii) take for A the poset of subgroups of a finite group, G, with reverse inclusion. The basic patches are stratified $\mathbb{R}^n \to A$, but can act as the basics for orbifold charts. (This needs more investigation and relating to other aspects of orbifolds, e.g. their relationship with groupoids. Another question is how the corresponding $(\infty, 1)$-category of exit paths, as studied by Lurie and by Ayala, Francis and Tanaka, relates to the the original data on the orbifold may be of use here.)

Problems and questions:

Produce Yetter-type models for orbifold TQFTs using stratified triangulations of stratified manifolds and cobordisms. (A start on something along these lines has been made by Doughterty, Park and Yetter.)

Produce a homotopy quantum field theory version of defects / singular TQFTs with coefficients in a stratified / filtered homotopy type (extending the idea of the thin filtration of $\overline{W}G$).
This may involve further investigation of the quasi-categorical / $(\infty, 1)$-categorical viewpoint introduced by Lurie, [2], and further used by Ayala et al. (Note our earlier question about replacing $\overline{W}(G)$ by a quasi-category.)

If that works try other forms of (∞, n)-category or A_∞-category, $\mathcal{C}$, in place of the homotopy type, ... but need finiteness conditions and there is the question of interpretation of the end results. What are $\mathcal{C}$-manifolds, or cobordisms?

That is unknown territory.

Thank you.

The End.

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