2-representations of finitary 2-categories

(joint work with Volodymyr Mazorchuk and Xiaoting Zhang)

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- natural (strict) axioms.

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- the full 2-subcategory $\mathfrak{A}_{\mathbb{k}}^{f}$ of $\mathfrak{A}_{\mathbb{k}}$ consisting of small idempotent complete $\mathbb{k}$-linear categories with finitely many indecomposable objects up to isomorphism and finite-dimensional morphism spaces (note such categories are equivalent to the category of finitely generated projectives over a finite-dimensional $\mathbb{k}$-algebra!).

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It is called **weakly fiat** if $(-)^*$ is not involutive.

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$\mathscr{C}_A$ is finitary. It is weakly fiat if A is self-injective (Frobenius), and fiat if A is weakly symmetric.

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Examples.

- The 2-category $\mathscr{C}_A$ was defined via its **defining** 2-representation.
- For each object $\mathtt{i}$ in a finitary 2-category $\mathscr{C}$, we have the **principal** 2-representation $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$.

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Cell 2-representations

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$\mathtt{i}$: the source for 1-morphisms in $\mathcal{L}$
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Simple transitive 2-representations and Jordan-Hölder Theorem

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Theorem. [Mazorchuk-M.] **M** has a **weak Jordan-Hölder filtration** with transitive subquotients $\mathbf{M}_j$. The list of simple transitive quotients of the $\mathbf{M}_j$ is unique up to permutation and equivalence.

Classification of simple transitive 2-representations?

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However, there exist simple transitive 2-representations that are not cell 2-representations, e.g. for Soergel bimodules in other types.