# Talk 1.3: Restriction species

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# Categorifying linear algebra

- We work in a monoidal closed $\infty$-category LIN
  - objects are all slices $\mathfrak{S}_{/I}$ of the $\infty$-category $\mathfrak{S}$ of $\infty$-groupoids,
  - morphisms are linear functors $\mathfrak{S}_{/I} \dashrightarrow \mathfrak{S}_{/J}$.
- A slice $\mathfrak{S}_{/B}$ should be thought of as a generalised 'vector space with specified basis': any $X \to B$ is a homotopy linear combination

$$\text{“}\sum_{b\in\pi_0 B} X_b \cdot (1 \xrightarrow{\ulcorner b \urcorner} B)\text{”} \;=\; \int^{b\in B} X_b \otimes \ulcorner b \urcorner \;:=\; \operatorname{hocolim}\begin{pmatrix} B \to \mathfrak{S} \\ b \mapsto X_b \end{pmatrix}.$$

- Any $f : B' \to B$ gives adjoint functors between slice categories

$$\mathfrak{S}_{/B'} \xrightarrow{f_!} \mathfrak{S}_{/B}, \qquad \mathfrak{S}_{/B} \xrightarrow{f^*} \mathfrak{S}_{/B'}$$

  defined by post-composition and homotopy pullback.
- Using these constructions, any span between $I$ and $J$

$$I \xleftarrow{p} M \xrightarrow{q} J$$

  defines a so-called linear functor

$$\mathfrak{S}_{/I} \xrightarrow{p^*} \mathfrak{S}_{/M} \xrightarrow{q_!} \mathfrak{S}_{/J}.$$

- Composition is '**matrix multiplication**': the Beck–Chevalley condition says the composite of linear functors defined by spans $I \leftarrow M \rightarrow J$ and $J \leftarrow N \rightarrow K$ is defined by the pullback span $I \leftarrow P \rightarrow K$:

$$\begin{tikzcd}
 & & K \\
 & P \arrow[r] \arrow[d] \arrow[ur] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & N \arrow[u] \arrow[d] \\
I & M \arrow[l] \arrow[r] & J
\end{tikzcd}$$

- The monoidal structure on LIN is defined on slices by

$$\mathfrak{S}_{/I} \otimes \mathfrak{S}_{/J} \ := \ \mathfrak{S}_{/I \times J}$$

with $\mathfrak{S} = \mathfrak{S}_{/1}$ as unit,

- and the tensor product of two linear functors defined by spans

$$I \leftarrow M \rightarrow J \quad \text{and} \quad K \leftarrow N \rightarrow L$$

is the linear functor defined by the span $I \times K \leftarrow M \times N \rightarrow J \times L$,

$$(\mathfrak{S}_{/I} \dashrightarrow \mathfrak{S}_{/J}) \otimes (\mathfrak{S}_{/K} \dashrightarrow \mathfrak{S}_{/L}) \ = \ (\mathfrak{S}_{/I \times K} \dashrightarrow \mathfrak{S}_{/J \times L})$$

# Cardinality of $\infty$-groupoids and linear functors

[Quinn (1995), Baez–Dolan (2001)]

- An $\infty$-groupoid $X$ is locally finite if $\forall x \in \pi_0 X$ the homotopy groups $\pi_i(X, x)$ are finite for $i \geq 1$ and are eventually trivial.
- A locally finite $\infty$-groupoid $X$ is finite if $\pi_0 X$ is finite.
- The cardinality of $X$ is then $|X| = \sum_{x_0 \in \pi_0 X} \prod_{i \geq 1} |\pi_i(X, x_0)|^{(-1)^i} \in \mathbb{Q}$
- The cardinality of $(X \to B) \in \mathfrak{S}_{/B}$ is the vector $\sum_{b \in \pi_0 B} |X_b| \, e_b \in \mathbb{Q}\,\pi_0 B$
- That of a span $S \leftarrow M \rightarrow T$ is a matrix $|M| : \mathbb{Q}\,\pi_0 S \to \mathbb{Q}\,\pi_0 T$
- The cardinality of a finite $\infty$-groupoid is

$$|X| \; := \; \sum_{x \in \pi_0 X} \prod_{i > 0} |\pi_i(X, x)|^{(-1)^i} \; \in \; \mathbb{Q}.$$

- In particular, we have equivalence-invariant notions of finiteness and cardinality of ordinary groupoids

$$|G| := \sum_{x \in \pi_0 G} \frac{1}{|\mathrm{Aut}_G(x)|}.$$

# $\infty$-categorification of the notion of coalgebra

A coalgebra in LIN is a slice $\mathfrak{S}_{/I}$ together with linear functors

$\mathfrak{S}_{/I} \overset{\delta_0}{\dashrightarrow} \mathfrak{S}$ (counit) and $\mathfrak{S}_{/I} \overset{\delta_2}{\dashrightarrow} {\mathfrak{S}_{/I}}^{\otimes 2} = \mathfrak{S}_{/I^2}$ (comultiplication)

that are counital:
$(1 \otimes \delta_0)\delta_2 = 1 = (\delta_0 \otimes 1)\delta_2$

$$\begin{tikzcd}
\mathfrak{S}^{\otimes 2}_{/I} \arrow[r, dashed, "1\otimes\delta_0"] & \mathfrak{S}_{/I} \\
\mathfrak{S}_{/I} \arrow[u, dashed, "\delta_2"] \arrow[ur, dashed, "1"] \arrow[r, dashed, "\delta_2"'] & \mathfrak{S}^{\otimes 2}_{/I} \arrow[u, dashed, "\delta_0\otimes 1"']
\end{tikzcd}$$

and coassociative:
$(1 \otimes \delta_2)\delta_2 = (\delta_2 \otimes 1)\delta_2$

$$\begin{tikzcd}
\mathfrak{S}^{\otimes 2}_{/I} \arrow[r, dashed, "1\otimes\delta_2"] & \mathfrak{S}^{\otimes 3}_{/I} \\
\mathfrak{S}_{/I} \arrow[u, dashed, "\delta_2"] \arrow[r, dashed, "\delta_2"'] & \mathfrak{S}^{\otimes 2}_{/I} \arrow[u, dashed, "\delta_2\otimes 1"']
\end{tikzcd}$$

Suppose the linear functors $\delta_0$ and $\delta_2$ are defined by the spans

$$I \xleftarrow{\;s\;} M \longrightarrow 1 \qquad\qquad I \xleftarrow{\;m\;} N \xrightarrow{\;c\;} I^2.$$

Then the counital and the coassociative properties can be written:

$$\begin{tikzcd}
I^2 & I \times M \arrow[l] \arrow[r] & I \\
N \arrow[u] \arrow[d] & I \arrow[l] \arrow[r] \arrow[u] \arrow[d] \arrow[ur, "1"] \arrow[dl, "1"'] \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & M \times I \arrow[u] \arrow[d] \\
I & N \arrow[l] \arrow[r] & I^2
\end{tikzcd}$$

$$\begin{tikzcd}
I^2 & I \times N \arrow[l] \arrow[r] & I^3 \\
N \arrow[u] \arrow[d] & P \arrow[l] \arrow[r] \arrow[u] \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & N \times I \arrow[u] \arrow[d] \\
I & N \arrow[l] \arrow[r] & I^2
\end{tikzcd}$$

# Recovering classical coalgebras

- Consider spans $I \xleftarrow{s} M \to 1, \quad I \xleftarrow{m} N \xrightarrow{c} I^2$ satisfying the above counit and coassociativity conditions and that restrict to linear functors on slices of the category $\mathfrak{s}$ of finite $\infty$-groupoids

**Theorem**

*Taking cardinality of such a finite coalgebra in LIN defines a classical coalgebra structure on the vector space $\mathbb{Q}\pi_0 I$*

$$
\begin{tikzcd}
\mathfrak{s}_{/I} \arrow[r, dashed] & \mathfrak{s} \\
\mathbb{Q}\pi_0 I \arrow[r] & \mathbb{Q}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{s}_{/I} \arrow[r, dashed] & {\mathfrak{s}_{/I}}^{\otimes 2} \\
\mathbb{Q}\pi_0 I \arrow[r] & \mathbb{Q}\pi_0 I^{\otimes 2}.
\end{tikzcd}
$$

# Incidence coalgebras in LIN

- For any simplicial $\infty$-groupoid $X$, the spans

$$X_1 \xleftarrow{s_0} X_0 \longrightarrow 1, \qquad X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$$

define linear functors, termed counit and comultiplication

$$\delta_0 : \mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}_{/1}, \qquad \delta_2 : \mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}_{/(X_1 \times X_1)}.$$

- We have seen that for coassociativity, for example, we need:

$$\begin{tikzcd}
X_1 & X_2 \arrow[l, "d_1"'] \arrow[r, "(d_2,d_0)"'] \arrow[rr, bend left, "d_2"] & X_1 \times X_1 \arrow[r, two heads] & X_1 \\
X_2 \arrow[u, "d_1"] \arrow[d, "(d_2,d_0)"'] & X_3 \arrow[l, "d_2"'] \arrow[r, "(d_3,d_0 d_0)"'] \arrow[rr, bend left, "d_3"] \arrow[u, "d_1"'] \arrow[d, "(d_2^2,d_0)"] \arrow[ur, phantom, very near start, "\urcorner"] \arrow[dl, phantom, very near start, "\llcorner"] & X_2 \times X_1 \arrow[u, "d_1 \times \mathrm{id}"'] \arrow[d, "(d_2,d_0)\times \mathrm{id}"] \arrow[r, two heads] & X_2 \arrow[u, "d_1"'] \\
X_1 \times X_1 & X_1 \times X_2 \arrow[l, "\mathrm{id}\times d_1"] \arrow[r, "\mathrm{id}\times(d_2,d_0)"'] & X_1 \times X_1 \times X_1 &
\end{tikzcd}$$

- This is equivalent to a certain other set of squares being pullbacks.

## Definition (Decomposition space [G-K-T, arXiv:1404.3202])

A decomposition space is a simplicial $\infty$-groupoid

$$X : \mathbb{\Delta}^{\mathrm{op}} \to \mathfrak{S}$$

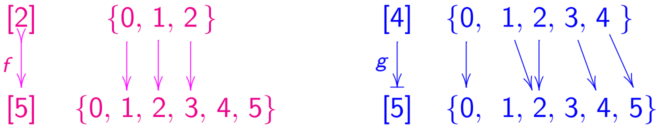
sending certain pushouts in $\mathbb{\Delta}$ to pullbacks in $\mathfrak{S}$

$$X\left(\begin{tikzcd} {[n]} \arrow[r, tail, "f"] \arrow[d, "g"'] & {[m]} \arrow[d, "g'"] \\ {[q]} \arrow[r, tail, "f'"'] & {[p]} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}\right) \quad = \quad \begin{tikzcd} X_p \arrow[r, "f'^*"] \arrow[d, "g'^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_q \arrow[d, "g^*"] \\ X_m \arrow[r, "f^*"'] & X_n. \end{tikzcd}$$

The pushouts involved are those for which
$g, g'$ are generic (that is, end-point preserving) maps in $\mathbb{\Delta}$,
$f, f'$ are free (that is, distance-preserving) maps in $\mathbb{\Delta}$.

This notion in fact coincides with that of unital 2-Segal space formulated independently by Dyckerhoff and Kapranov, see arXiv:1212.3563, arXiv:1306.2545, arXiv:1403.5799.

- Free maps are composites of outer coface maps $\partial^\perp = \partial^0, \partial^\top = \partial^n$, generic maps are composites of inner coface & codegeneracy maps.




- There is a monoidal structure (on the generic subcategory)
$$[n] \vee [m] = [n+m].$$
- Free maps in $\mathbb{\Delta}$ are the 'obvious' inclusions $[n] \rightarrowtail [a] \vee [n] \vee [b]$.

**Lemma**

*Generic and free maps in $\mathbb{\Delta}$ admit pushouts along each other, and the results are again generic and free.*

$$\begin{tikzcd}
{[n]} \arrow[r, tail, "f"] \arrow[d, "g"'] & {[a] \vee [n] \vee [b] = [a+n+b]} \arrow[d, "id \vee g \vee id"] \\
{[q]} \arrow[r, tail, "f'"] & {[a] \vee [q] \vee [b] = [a+q+b]} \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

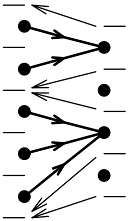
These are the pushouts that any decomposition space $X : \mathbb{\Delta}^{op} \to \mathfrak{S}$ is required to send to pullbacks of $\infty$-groupoids.

# Simplicial category $\mathbb{\Delta}$ v augmented simplicial category $\underline{\mathbb{\Delta}}$

- The objects of $\mathbb{\Delta}$ are denoted $[n] := \{0, 1, \ldots, n\}$, $\qquad n \geq 0$.
  The monotone maps are generated by
  - $s^k : [n{+}1] \to [n]$ that repeats the element $k \in [n]$,
  - $d^k : [n] \to [n{+}1]$ that skips the element $k \in [n{+}1]$.
- The objects of $\underline{\mathbb{\Delta}}$ are denoted $\underline{n} := \{1, \ldots, n\}$, $\qquad n \geq 0$.
  The monotone maps are generated by
  - $\underline{s}^k : \underline{n{+}1} \to \underline{n}$ that repeats the element $k+1 \in \underline{n}$, $(0 \leq k \leq n-1)$,
  - $\underline{d}^k : \underline{n} \to \underline{n{+}1}$ that skips the element $k+1 \in \underline{n{+}1}$, $(0 \leq k \leq n)$.
- There is a canonical contravariant isomorphism of categories between the generic subcategory of $\mathbb{\Delta}$ and the augmented simplicial category.

[Joyal–Stone duality] $\qquad {\mathbb{\Delta}_{\mathrm{gen}}}^{\mathrm{op}} \cong \underline{\mathbb{\Delta}}$

- the degeneracy map $\quad s^k : [n{+}1] \to [n]$
  corresponds to $\quad \underline{d}^k : \underline{n} \to \underline{n{+}1}$
- an inner coface map $d^{k+1} : [n] \to [n{+}1]$
  corresponds to $\quad \underline{s}^k : \underline{n{+}1} \to \underline{n}$



Picture: a map $[5] \leftarrow [4]$ in $\mathbb{\Delta}_{\mathrm{gen}}$ and $\underline{5} \to \underline{4}$ in $\underline{\mathbb{\Delta}}$ :

# Conservative and ULF maps

A simplicial map $F : Y \to X$ is called

- cartesian on a generic map $g\colon [m] \twoheadrightarrow [n]$ in $\mathbb{\Delta}$ if the naturality square

$$\begin{tikzcd} Y_m \arrow[d, "F_m"'] & Y_n \arrow[l, "g^*"'] \arrow[d, "F_n"] \arrow[dl, phantom, "\lrcorner"] \\ X_m & X_n \arrow[l, "g^*"] \end{tikzcd}$$

  is a pullback.

- conservative if $F$ is cartesian on all codegeneracy maps $\sigma^i_n$ of $\mathbb{\Delta}$,
- ULF if $F$ is cartesian on generic coface maps $\partial^i_n$, $i \neq 0, n$, of $\mathbb{\Delta}$,
- cULF (that is, conservative with Unique Lifting of Factorisations) if it is cartesian on all generic maps of $\mathbb{\Delta}$.

Such maps behave well on decomposition spaces. For example:

**Lemma**

If $F$ is cULF and $X$ is a decomposition space then so is $Y$.

# The incidence coalgebra of a decomposition space

- Let $X$ be a decomposition space. For $n \geq 0$ there is a linear functor
$$\delta_n : \mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}_{/X_1} \otimes \cdots \otimes \mathfrak{S}_{/X_1}$$
termed the $n$th comultiplication map, defined by the span
$$X_1 \longleftarrow X_n \longrightarrow X_1 \times \cdots \times X_1$$
- $\delta_0$ is the counit, and $\delta_1$ is the identity.

**Theorem (Coherent coassociativity)**

Any linear functor $\mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}_{/X_1} \otimes \cdots \otimes \mathfrak{S}_{/X_1}$
given by composites of tensors of comultiplication maps
is again a comultiplication map.

In particular, $(1 \otimes \delta_0)\delta_2 = 1 = (\delta_0 \otimes 1)\delta_2, (1 \otimes \delta_2)\delta_2 = \delta_3 = (\delta_2 \otimes 1)\delta_2$
so $C(X) := \mathfrak{S}_{/X_1}$ is a (counital, coassociative) coalgebra in LIN.

# Functoriality for cULF maps of decomposition spaces

Recall that a map $F : X \to X'$ of simplicial groupoids is said to be conservative and ULF (cULF) if it is cartesian on generic maps.

$$
\begin{tikzcd}
X_1 \arrow[d, "F_1"'] & X_n \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "F_n"] \arrow[dl, phantom, very near start, "\llcorner"] & X_1^n \arrow[d, "F_1^n"] \\
X_1' & X_n' \arrow[l, "g'"] \arrow[r, "f'"'] & {X_1'}^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{S}_{/X_1} \arrow[r, "g^*"'] \arrow[d, "F_{1!}"'] \arrow[rr, dashed, bend left, "\delta_n"] & \mathfrak{S}_{/X_n} \arrow[r, "f_!"'] \arrow[d, "F_{n!}"'] & \mathfrak{S}^{\otimes n}_{/X_1} \arrow[d, "{F_{1!}}^{\otimes n}"] \\
\mathfrak{S}_{/X_1'} \arrow[r, "g'^*"] \arrow[rr, dashed, bend right, "\delta_n'"'] & \mathfrak{S}_{/X_n'} \arrow[r, "f'_!"] & \mathfrak{S}^{\otimes n}_{/X_1'}
\end{tikzcd}
$$

Thus any cULF map $F : X \to X'$ between decomposition spaces induces a homomorphism of coalgebras $F_{1!} : C(X) \to C(X')$, since $F_{1!} : \mathfrak{S}_{/X_1} \to \mathfrak{S}_{/X_1'}$ commutes with comultiplication maps.

# Decalage

Recall that the augmented functors Dec$^\perp$ and Dec$^\top$ forget the bottom and top face and degeneracy maps respectively.

$$
\begin{tikzcd}
X \\
\\
\mathrm{Dec}^\perp X \arrow[uu, "d_\perp"]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
X_0 \arrow[r, "s_0" description]
& X_1 \arrow[l, shift right=3, "d_1"'] \arrow[l, shift left=3, "d_0"] \arrow[r, shift left=2, "s_1" description] \arrow[r, shift right=2, "s_0" description]
& X_2 \arrow[l, shift right=4, "d_2"'] \arrow[l, "d_1" description] \arrow[l, shift left=4, "d_0"] \arrow[r, shift left=4, "s_2" description] \arrow[r, "s_1" description] \arrow[r, shift right=4, "s_0" description]
& X_3 \arrow[l, shift right=6, "d_3" description] \arrow[l, shift right=2, "d_2" description] \arrow[l, shift left=2, "d_1" description] \arrow[l, shift left=6, "d_0" description]
& \cdots \\
X_1 \arrow[u, "d_0"] \arrow[r, "s_1" description]
& X_2 \arrow[u, "d_0"] \arrow[l, shift right=3, "d_2"'] \arrow[l, shift left=3, "d_1"] \arrow[r, shift left=2, "s_2" description] \arrow[r, shift right=2, "s_1" description]
& X_3 \arrow[u, "d_0"] \arrow[l, shift right=4, "d_3"'] \arrow[l, "d_2" description] \arrow[l, shift left=4, "d_1"] \arrow[r, shift left=4, "s_3" description] \arrow[r, "s_2" description] \arrow[r, shift right=4, "s_1" description]
& X_4 \arrow[u, "d_0"] \arrow[l, shift right=6, "d_4" description] \arrow[l, shift right=2, "d_3" description] \arrow[l, shift left=2, "d_2" description] \arrow[l, shift left=6, "d_1" description]
& \cdots
\end{tikzcd}
$$

**Lemma**

*A simplicial $\infty$-groupoid $X : \mathbb{\Delta}^{op} \to \mathfrak{S}$ is a decomposition space if and only if both* Dec$^\top(X)$ *and* Dec$^\perp(X)$ *are Segal spaces, and the two comparison maps are cULF:*

$$d_\top : \mathrm{Dec}^\top(X) \to X \qquad\qquad d_\perp : \mathrm{Dec}^\perp(X) \to X$$

# Monoidal decomposition spaces

- Recall that a bialgebra is a coalgebra with a compatible algebra structure, i.e. multiplication and unit are coalgebra homomorphisms.
- A simplicial map $f$ between decomposition spaces induces a coalgebra homomorphism on incidence coalgebras if $f$ is cULF.
- Accordingly we define a monoidal decomposition space to be a decomposition space $Z$ equipped with an associative unital multiplication given by cULF maps $m : Z \times Z \to Z$ and $e : 1 \to Z$.

If $Z$ is a monoidal decomposition space then $C(Z) = \mathfrak{S}_{/Z_1}$ is naturally a bialgebra, termed its *incidence bialgebra*.

- Its cardinality inherits a classical bialgebra structure.
- cULF monoidal maps between monoidal decomposition spaces induce bialgebra maps.

The Dec of a monoidal decomposition space has again a natural monoidal structure, and the dec map is cULF monoidal.

# A classical example

- Let $\mathbb{B}$ be the monoidal groupoid of finite sets and bijections, with monoidal structure given by disjoint union, and let $\mathbf{B}$ be its nerve.
- Let $\mathbb{I}$ be the category of finite sets and injections, with nerve $\mathbf{I}$.
- Dür (1985): on identifying injections with isomorphic complements in the incidence coalgebra of $\mathbb{I}$, one obtains the binomial coalgebra.
- In our language: the lower dec map gives a conservative ULF functor

$$\mathbf{I} \qquad \cong \qquad \mathrm{Dec}_\perp(\mathbf{B}) \xrightarrow{d_\perp} \mathbf{B}$$

$$(x_0 \subseteq x_0{+}x_1 \subseteq \cdots \subseteq x_0{+}\cdots x_k) \leftarrow\!\shortmid (x_0, x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k)$$

- In terms of Waldhausen's $S_\bullet$-construction, $d_\perp$ deletes the last row. For $k = 3$:

$$\begin{tikzcd}
 & & & x_3 \arrow[d] \\
 & & x_2 \arrow[r] \arrow[d] & x_2 + x_3 \arrow[d] \\
 & x_1 \arrow[r] \arrow[d] & x_1 + x_2 \arrow[r] \arrow[d] & x_1 + x_2 + x_3 \arrow[d] \\
x_0 \arrow[r] & x_0 + x_1 \arrow[r] & x_0 + x_1 + x_2 \arrow[r] & x_0 + x_1 + x_2 + x_3
\end{tikzcd}$$

so that the effect on a chain of injections

$$[x_0 \subseteq x_0 + x_1 \subseteq x_0 + x_1 + x_2 \subseteq \cdots \subseteq x_0 + x_1 \cdots + x_k]$$

is to send it to the sequence of successive complements of inclusions

$$[x_1, x_2, \ldots, x_k]$$

- Both $\mathbf{I}$ and $\mathbf{B}$ are monoidal decomposition spaces under disjoint union, and this comparison functor is monoidal also, inducing a quotient homomorphism of incidence bialgebras

$$C(\mathbf{I}) \to C(\mathbf{B})$$

# Numerical section coefficients for incidence coalgebras

If $X$ is a locally finite decomposition space, the cardinality of

$$\delta_2 : \mathfrak{s}_{/X_1} \dashrightarrow \mathfrak{s}_{/X_1} \otimes \mathfrak{s}_{/X_1}$$

may be written in terms of the canonical basis $e_f = |1 \xrightarrow{f} X_1|$ as

$$\mathbb{Q}\pi_0 X_1 \longrightarrow \mathbb{Q}\pi_0 X_1 \otimes \mathbb{Q}\pi_0 X_1, \qquad e_f \mapsto \sum_{a,b} c^f_{a,b}\, e_a \otimes e_b.$$

Here $c^f_{a,b}$ is given by cardinalities of components of $X_1$ and of fibres of face maps $d_1, d_0, d_2 : X_2 \to X_1$.

$$c^f_{a,b} = \left|(X_1)_{[a]}\right| \left|(X_1)_{[b]}\right| \left|(X_2)_{f,a,b}\right|.$$

# The “linear dual” of the incidence coalgebra

## The incidence algebra of linear functors $\mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}$

- If $X$ is a decomposition space, the linear functors $\mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}$ form the convolution algebra, dual to the incidence coalgebra $C(X)$.
- Its cardinality is the classical convolution algebra $\mathbb{Q}^{\pi_0 X_1}$, if $X$ is locally finite, that is dual to the classical incidence coalgebra.
- Let $F, G$ be defined by spans $X_1 \leftarrow A \rightarrow 1$ and $X_1 \leftarrow B \rightarrow 1$. Their convolution is $F * G = (F \otimes G)\, \delta_2$, defined by the span

$$
\begin{tikzcd}
 & & 1 \\
 & A * B \arrow[r] \arrow[d] \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & A \times B \arrow[u] \arrow[d] \\
X_1 & X_2 \arrow[l] \arrow[r] & X_1 \times X_1
\end{tikzcd}
$$

# The Zeta functor and Möbius inversion

- The counit $\delta_0 : \mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}$ is neutral for convolution.
- The zeta functor $\zeta : \mathfrak{S}_{/X_1} \dashrightarrow \mathfrak{S}$ is the linear functor defined by the span $X_1 \xleftarrow{=} X_1 \longrightarrow 1$.

The zeta functor has convolution-inverse, the Möbius functor, except for the lack of additive inverses.

- The convolution inverse to $\zeta$ should be the Möbius functor:

$$\text{“}\mu = \mu_{\text{even}} - \mu_{\text{odd}}\text{”}, \qquad \text{“}\zeta * (\mu_{\text{even}} - \mu_{\text{odd}}) = \delta_0\text{”},$$

Before taking cardinality we have no negative quantities, but we can define linear functors $\mu_{\text{even}}, \mu_{\text{odd}}$ with

$$\zeta * \mu_{\text{even}} = \delta_0 + \zeta * \mu_{\text{odd}}. \qquad (\star)$$

# Completeness

- The axioms of Lawvere-Menni for Möbius categories ensure that the Möbius inversion formula is a finite sum of terms: they say that an arrow can be factored only in finitely many ways as a chain of non-identity arrows.
- In the simplicial nerve $X$ of a Möbius category, this says that there are only finitely many non-degenerate simplices whose long edge is a given arrow $a \in X_1$.
- For a general simplicial object, degenerate should mean to be in the 'image' of the degeneracy maps. What about non-degenerate?
- The 'complement of the image' makes sense here if the degeneracy maps $s_i : X_n \to X_{n+1}$ are fully faithful as functors of $\infty$-groupoids, that is, maps whose homotopy fibres are empty or contractible.
- For decomposition spaces, the case $s_0 : X_0 \to X_1$ is enough.

- If $X$ is a decomposition space in which $s_0 : X_0 \to X_1$ is fully faithful, we define linear functors $\mu_r$ by the spans

$$X_1 \xleftarrow{d_1^{r-1}} \vec{X}_r \longrightarrow 1.$$

Here $\vec{X}_r$ is the subgroupoid of non-degenerate simplices.

**Theorem (Möbius inversion without additive inverses)**

*The linear functors $\mu_r$ satisfy*

$$\mu_0 = \delta_0, \qquad \zeta * \mu_r \;=\; \mu_r + \mu_{r+1}.$$

*Thus*

$$\mu_{even} := \sum_{r \ even} \mu_r, \quad \mu_{odd} := \sum_{r \ odd} \mu_r$$

*satisfy*

$$\zeta * \mu_{even} = \delta_0 + \zeta * \mu_{odd}. \qquad (\star)$$

# Restriction species

- A restriction species (Schmitt, 1993) is a presheaf on the category of finite sets and injections.

$$R : \mathbb{I}^{op} \to \underline{\mathbf{Set}}$$

$$S \mapsto R[S]$$

- Recall: a (classical) species is a presheaf on finite sets and bijections.
- An $R$-structure on a finite set $S$ restricts to one on each of its subsets $A$ (whence the name) denoted by a restriction bar:

$$A \subset S$$

$$R[S] \to R[A]$$

$$X \mapsto X|A$$

# Coalgebras from restriction species

Schmitt associates to a restriction species a coalgebra spanned by isoclasses of $R$-structures, with the comultiplication defined by

$$\delta_2(X) = \sum_{A+B=S} X|A \otimes X|B \qquad X \in R[S]$$

and the counit $\varepsilon$ sending the empty $R$-structure to 1.

# Schmitt's graph coalgebra

- For any set $V$, consider the set of graphs $G$ with vertex set $V$.
- For $U \subseteq V$, $G|U$ is the graph restricted to the vertex set $U$.
- This is clearly a restriction species.
- Its associated coalgebra is Schmitt's graph coalgebra.
- In fact, it is the cardinality of a coalgebra associated to a decomposition space $\mathbf{G}$.

# The decomposition space **G**

**G** is the simplicial groupoid with

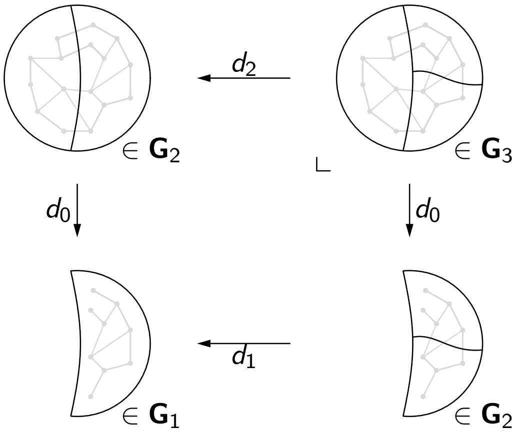
- $\mathbf{G}_0$ a point (a contractible groupoid) labelled by the empty graph
- $\mathbf{G}_1$ the groupoid of all graphs and their isomorphisms
- $\mathbf{G}_k$ the groupoid of graphs endowed with an ordered partition of their vertex sets $V(G)$ into $k$ possibly empty parts

and simplicial structure given as follows:

- The degeneracies $s_j$ insert an empty $(j+1)$st part
- The inner faces $d_i$, for $0 < i < n$, join adjacent parts $i, i+1$
- The outer faces $d_0, d_n$ delete the first, last part of the graph

**G** is not a Segal space (can't reconstruct a graph just from its parts)
It is a decomposition space (pictorial proof on next slide!)
and its cardinality is Schmitt's coalgebra of graphs.

# Simplicial structure, and the decomposition space axiom




The horizontal maps join two parts of the vertex partition.
The vertical maps forget a part (and any incident edges).
Pullback condition: a graph with a 3-part partition of its vertices (top right) can be reconstructed uniquely from the other data.

# Restriction species to decomposition spaces to coalgebras

- For any $R : \mathbb{I}^{op} \to \underline{\mathbf{Grpd}}$, let $\mathbb{R} \to \mathbb{I}$ be the Grothendieck construction (of all $R$-structures and their $R$-structure preserving injections).
- Each fibre $\mathbb{R}_k$ is the groupoid of all $R$-structures with an ordered partition of the underlying set into $k$ possibly empty parts.
- $\mathbb{R}$ defines a simplicial groupoid $\mathbf{R}$ which is a decomposition space
- The cardinality of $C(\mathbf{R})$ is Schmitt's coalgebra of $R$.
- Restriction species maps induce cULF decomposition space maps, and homomorphisms of their coalgebras, and of their cardinalities.
- Examples of restriction species are many:
  several classes of graphs, matroids, posets, simplicial complexes, ...

# Directed restriction species

The category $\mathbb{C}$ of posets and convex maps

- A poset map $f : K \to P$ is convex if for all $a, b \in K$ and $fa \leq x \leq fb$ in $P$ there is a unique $k \in K$ with $a \leq k \leq b$ and $fk = x$.
- That is, $f$ is injective, and the image $f(K) \subset P$ is a convex subposet.
- We denote by $\mathbb{C}$ the category of finite posets and convex maps.
- Convex maps are stable under pullback.
- An $n$-layering is a poset map $\ell : P \to \underline{n}$, where $\underline{n} = \{1, 2, \ldots, n\}$.
- The layers $P_i = \ell^{-1}(i)$ are convex subposets (and may be empty).
- Consider the groupoid $\mathbb{C}^{iso}_{/\underline{n}}$ of all $n$-layerings of finite posets.

  Objects: poset maps $\ell : P \to \underline{n}$

  Morphisms: triangles

$$\begin{tikzcd} P \arrow[rr, "\cong"] \arrow[dr, "\ell"'] & & P' \arrow[dl, "\ell'"] \\ & \underline{n}, & \end{tikzcd}$$

# Directed restriction species

## The decomposition space $\mathbb{C}$ of layered finite posets

- We can define simplicial maps between the groupoids of layered finite posets so they form a decomposition space $\mathbf{C} = \{\mathbb{C}^{\mathrm{iso}}_{/\underline{n}}\}$.
- If $g : [n] \to [m]$ is a generic map in $\mathbb{\Delta}$, and if $\underline{g} : \underline{m} \to \underline{n}$ is the corresponding map in $\underline{\mathbb{\Delta}}$, then $g^* : \mathbb{C}^{\mathrm{iso}}_{/\underline{m}} \to \mathbb{C}^{\mathrm{iso}}_{/\underline{n}}$ is postcomposition:

$$P \to \underline{m} \qquad \mapsto \qquad P \to \underline{m} \xrightarrow{\underline{g}} \underline{n}.$$

- If $f : [n] \to [m]$ is a free map, $f^*$ is defined by pullback. e.g. $d_\top : \mathbb{C}^{\mathrm{iso}}_{/\underline{n}} \to \mathbb{C}^{\mathrm{iso}}_{/\underline{n-1}}$ takes $P \to \underline{n}$ to $P' \to \underline{n-1}$ in the pullback

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P \arrow[d] \\ \underline{n-1} \arrow[r, "\underline{d}_\top"'] & \underline{n}. \end{tikzcd}$$

# Directed restriction species

- A directed restriction species is a functor $R : \mathbb{C}^{op} \to \underline{\mathbf{Grpd}}$ (that is, a right-fibration $\mathbb{R} \to \mathbb{C}$ of all $R$-structures on finite posets.)
- A 2-layering of a poset $P$ defines an admissible cut of each $X \in R[P]$.
- The incidence coalgebra of $R$ is the vector space spanned by isomorphism classes in $\mathbb{R}$, and one defines the comultiplication

$$\delta_2(X) = \sum X|D_c \otimes X|U_c, \qquad X \in R[P],$$

  summing over all admissible cuts $(D_c, U_c)$ of the $R$-structure $X$ on $P$.
- The groupoids $\mathbb{R}_n$, of $R$-structures on posets with an $n$-layering, form a decomposition space $\mathbf{R}$ with a cULF map to $\mathbf{C}$.
- The cardinality of $C(\mathbf{R})$ is the incidence coalgebra of $R$.

# The decalage gives the nerve of a category

- Consider the category of finite posets, but with only the upper- (or lower-) set inclusions, rather than all convex maps

$$\mathbb{C}^{\text{lower}} \to \mathbb{C} \leftarrow \mathbb{C}^{\text{upper}}$$

- For each directed restriction species $R$, we obtain categories

$$\mathbb{R}^{\text{lower}} := \mathbb{C}^{\text{lower}} \times_{\mathbb{C}^{\text{iso}}} \mathbb{R}^{\text{iso}} \qquad \mathbb{R}^{\text{upper}} := \mathbb{C}^{\text{upper}} \times_{\mathbb{C}^{\text{iso}}} \mathbb{R}^{\text{iso}}$$

  of $R$-structures and their lower-set and upper-set inclusions.
- There are natural (levelwise) equivalences of simplicial groupooids

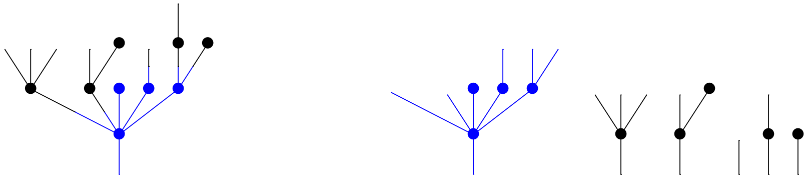
$$\mathrm{Dec}_\perp \mathbf{R} \simeq \mathbf{N}\mathbb{R}^{\text{lower}} \qquad \mathrm{Dec}_\top \mathbf{R} \simeq \mathbf{N}(\mathbb{R}^{\text{upper}})^{\text{op}}$$

# Monoidal (directed) restriction species

- The categories $\mathbb{I}$ and $\mathbb{C}$ are symmetric monoidal under disjoint union.
- A monoidal (directed) restriction species is a (directed) restriction species in which $\mathbb{R}$ has a monoidal structure and the map $\mathbb{R} \to \mathbb{I}$ (or $\mathbb{R} \to \mathbb{C}$) is strong monoidal.
- The functor from restriction species (or from directed restriction speacies) to decomposition spaces extends to a functor from monoidal (directed) restriction species and their morphisms, to monoidal decomposition spaces and cULF monoidal functors.
- It follows if a (directed) restriction species is monoidal, the associated incidence coalgebra becomes a bialgebra.
- As $\mathbb{R} \to \mathbb{C}$ is monoidal, the incidence bialgebra of a directed restriction species $\mathbb{R}$ comes with a bialgebra homomorphism to the incidence bialgebra of $\mathbb{C}$.

# Example: The Connes–Kreimer bialgebra

See also: I. Gálvez, J. Kock, A. Tonks, *Groupoids and Faà di Bruno formulae for Green functions in bialgebras of trees*. Advances in Mathematics 254 (2014) 79-117



$c : D_c \rightarrowtail T \qquad D_c, \qquad U_c\,.$

- A bialgebra studied by Dür (1986), and also by Butcher (1972).
- It has basis the isomorphism classes of forests, with multiplication given by disjoint union of forests, and comultiplication

$$\begin{array}{rcc} \delta_2 : \mathscr{B} & \longrightarrow & \mathscr{B} \otimes \mathscr{B} \\ T & \mapsto & \sum_c D_c \otimes U_c, \end{array}$$

where the sum is over all admissible cuts of $T$.

# Cancellation

Often a more economical Möbius function can be found for a decomposition space $X$, and be exploited to yield more economical formulae for any decomposition space with a cULF functor to $X$.

**Lemma**

*Suppose that for the complete decomposition space $X$ we have found*

$$\zeta * \Psi_0 = \zeta * \Psi_1 + \epsilon.$$

*Then for every decomposition space cULF over $X$, say $f : Y \to X$,*

$$\zeta * f^* \Psi_0 = \zeta * f^* \Psi_1 + \epsilon$$

*is a Möbius inversion formula for $Y$.*

This happens because convolution, $\epsilon$ and $\zeta$ are all preserved under precomposition with cULF maps.

# Decomposition spaces over $\mathbf{B}$

If a decomposition space $X$ admits a cULF functor $\ell : X \to \mathbf{B}$ (which may be thought of as a 'length function with symmetries') then at the numerical level and at the objective level we can pull back the economical Möbius 'functor' $\mu(n) = (-1)^n$ that exists for $\mathbf{B}$, yielding the numerical Möbius function on $X$

$$\mu(f) = (-1)^{\ell(f)}.$$

An example of this is the coalgebra of graphs of Schmitt:
the functor from the decomposition space of graphs to $\mathbf{B}$ which to a graph associates its vertex set is cULF.
Hence the Möbius function for this decomposition space is

$$\mu(G) = (-1)^{\# V(G)}.$$

In fact this argument works for any restriction species.

Thank you for your attention

Thank you for your attention

# Cancelation

# The simplicial category

- Denote by $\Delta$ the category whose objects $[n]$, $n \geq 0$, are finite nonempty standard ordinals (with $n+1$ elements!)

$$[n] := \{0, 1, \cdots, n\}, \qquad n \geq 0.$$

  and whose morphisms are the order-preserving maps between them.
- Among these maps one considers the following generators: for $n \geq 0$, $i \in [n]$,

$$\partial_n^i : [n-1] \to [n]$$

  is the unique nondecreasing injection that skips the value $i$, and

$$\sigma_n^i : [n+1] \to [n]$$

  is the unique nondecreasing surjection that repeats the value $i$.
- If there is no ambiguity, one writes $\partial^i$ and $\sigma^i$.

# Simplicial groupoids

- A simplicial groupoid is a functor

$$X : \Delta^{\mathrm{op}} \to \text{Groupoids}.$$

  One writes

$$\begin{aligned} X_n &\text{ for } X([n]) \\ d_i &\text{ for } X(\partial^i) : X_n \to X_{n-1}, \quad i = 0, \dots, n, \\ s_i &\text{ for } X(\sigma^i) : X_n \to X_{n+1}, \quad i = 0, \dots, n, \end{aligned}$$

- A simplicial map $X \to Y$ is a natural transformation: a family of maps $(X_n \to Y_n)_{n \geq 0}$ commuting with face and degeneracy maps.

# Infinity groupoids

Joyal: *Quasi-categories and Kan complexes* (2002), Lurie: *Higher Topos Theory* (2009)

- We often use simplicial $\infty$-groupoids instead of simplicial groupoids.
- One model for the notion of $\infty$-groupoid is that of Kan complex.
- The corresponding model for the notion of $\infty$-category is that of weak Kan complex, termed quasi-category by Joyal.
- Recall that the representable simplicial set $\mathbb{\Delta}[n]$ is the simplicial set defined by

$$\mathbb{\Delta}[n] := \mathrm{Hom}_{\mathbb{\Delta}}(-,[n]) : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$$

  and that its $k$-th horn $\Lambda^k[n]$ is the largest simplicial subset of $\mathbb{\Delta}[n]$ that does not contain either $1_{[n]} : [n] \to [n]$ or $\partial^k : [n-1] \to [n]$.
- A weak Kan complex is a simplicial set $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ with the Kan extension property along horn inclusions $\jmath^n_k : \Lambda^k[n] \to \mathbb{\Delta}[n]$ for $0 < k < n$, while a Kan complex has this extension property for all $0 \le k \le n$.

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "\forall"] \arrow[d, hook, "\jmath^n_k"'] & X \\
\mathbb{\Delta}[n] \arrow[ur, dotted, "\exists"'] &
\end{tikzcd}$$